Constant Flux Relation in Non-equilibrium Statistical Mechanics

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Joint work with:

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Some Terminology from Turbulence Theory

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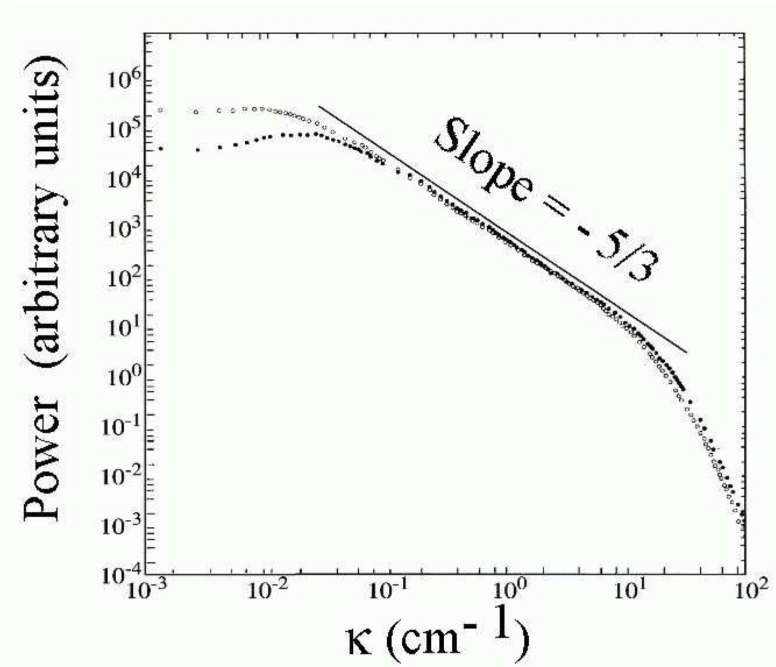


- Reynolds number $R = \frac{LU}{\nu}$.
- Energy injected into large eddies.
- Energy removed from small eddies at viscous scale.
- Transfer by interaction between eddies.
- Concept of *inertial range*

K41 : In the limit of ∞ R, all small scale statistical properties depend only on the local scale, k, and the energy dissipation rate, P. Dimensional analysis :

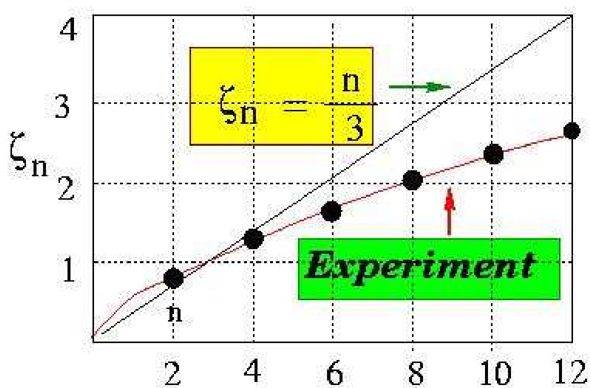
$$E(k) = cP^{\frac{2}{3}}k^{-\frac{5}{3}}$$ Kolmogorov spectrum

Structure Functions and the $\frac{4}{5}$ Law

- Structure functions : $S_n(r) = \langle (u(x+r) - u(x))^n \rangle$.
- Scaling form in stationary state:

$$\lim_{r\to 0} \lim_{\nu\to 0} \lim_{t\to\infty} S_n(r) = C_n \, (Pr)^{\zeta_n} \, .$$

- K41 theory gives $\zeta_n = \frac{n}{3}$.



n	n/3	ζ_n
2	0.67	0.70
4	1.33	1.20
6	2.00	1.62
8	2.67	2.00
10	3.33	2.36
12	4.00	2.68

$\frac{4}{5}$ Law : $S_3(r) = \frac{4}{5} Pr$. Thus $\zeta_3 = 1$ (exact for all d's).

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- Existence of a quantity, I, such as energy which is conserved by the nonlinear interactions in the system.

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Questions : Is Kolmogorov phenomenology useful for studying such systems? Is there a counterpart of the $4/5$ law?

Mass Aggregation Model.

Consider a lattice in d dimensions with massive point particles.
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Interested in the probability $P_1(m,t)$ of finding a particle at a site:

$$\begin{aligned}\frac{\partial P_1(m)}{\partial t} &= D\Delta P_1(m) + \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P_2(m_1, m_2, +0)\delta(m - m_1 - m_2) \\ &- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P_2(m, m_1, +0)\delta(m_2 - m - m_1) \\ &- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P(m, m_2, +0)\,\delta(m_1 - m_2 - m) + \frac{J}{m}\delta(m - m_0).\end{aligned}$$

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- Scaling of multi-point correlation functions in mass space?

$$P_n(m_1, \ldots, m_n, +0) = Pr.(n\, particles\, are\, in\, dV\, dm_1 \ldots dm_n).$$

Answers in $d < 2$

The critical dimension for the mass aggregation model is 2. Mean field scaling is correct for $d > 2$ but incorrect for $d \leq 2$. In $d = 2$ scaling laws acquire logarithmic corrections.

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Corrections to scaling exponents in $d < 2$ can be calculated using an ϵ-expansion where $\epsilon = 2 - d$.
It is found that (Phys. Rev. Lett. 94, 194503 (2005))

$$P_1(m) \sim m^{-\frac{2d+2}{d+2}}$$

and

$$P_n(m_1, \ldots, m_n) \sim m^{-\gamma_n} \Phi\left(\frac{m_i}{m_j}\right)$$

where

$$\gamma_n = \frac{3}{2}n + \frac{n(n-2)}{2(d+2)}\epsilon + O(\epsilon^2).$$

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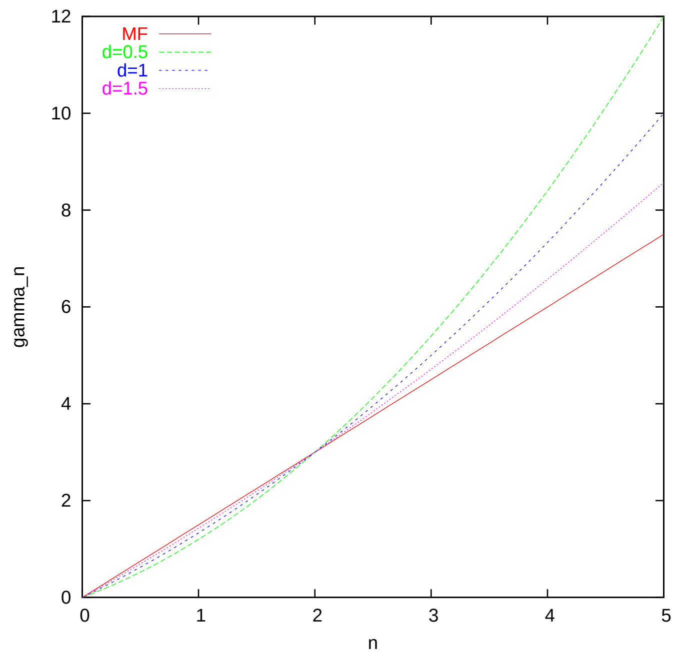

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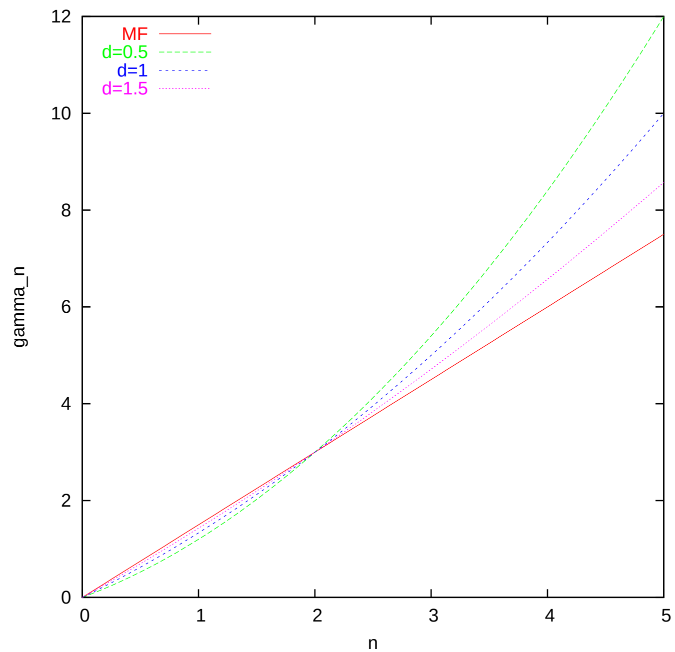

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- γ_1 can be obtained from a dimensional arguments if one assumes that P_1 depends on mass m and mass flux J only.
- Order ϵ correction to M.F. vanishes for 2-point function. Numerics suggested $\gamma_2 = 3$ exactly. Why?

Can we find γ_2 without recourse to ϵ-expansion?

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- Stationary Hopf equation for $m > m_0$:

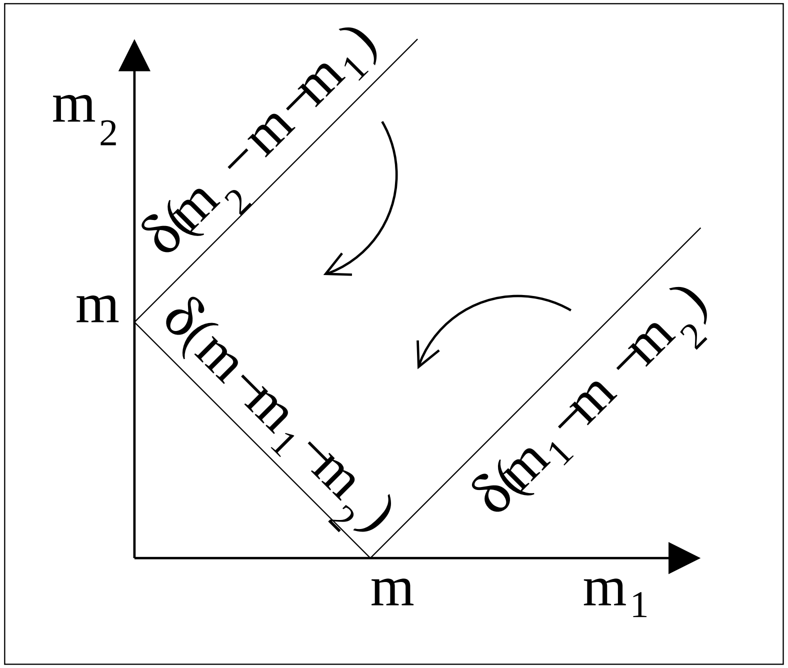
$$
\begin{aligned}
0 &= \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P_2(m_1, m_2)\, \delta(m - m_1 - m_2) \\
&- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P_2(m, m_1)\, \delta(m_2 - m - m_1) \\
&- \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, P_2(m, m_2)\, \delta(m_1 - m_2 - m).
\end{aligned}
$$

- Assume scaling form for 2-point function :

$$
P_2(m_1, m_2) = \frac{1}{(m_1 m_2)^h}\phi\left(\frac{m_1}{m_2}\right)
$$

- ϕ is a scaling function: $\phi(x) = \phi(1/x)$.

Zakharov Transformations



Change variables in the 2nd and 3rd integrals in Hopf equation

$$(m_1, m_2) \quad \rightarrow \quad \frac{m}{m_2'}(m_1', m)$$

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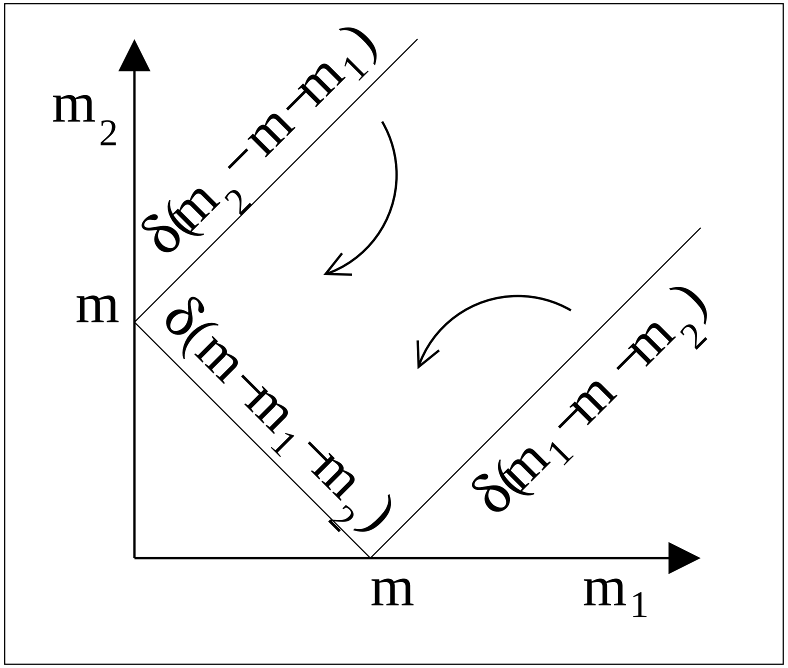

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$$
0 = \frac{\lambda}{2}\int_0^\infty dm_1 dm_2\, (m_1 m_2)^{-h} \phi\left(\frac{m_1}{m_2}\right) m^{2-2h} \left(m^{2h-2} - m_1^{2h-2} - m_2^{2h-2}\right)\, \delta(m - m_1 - m_2)
$$

Constant Flux Relation for the Mass Aggregation Model

- After Z.T. we see that the RHS of Hopf equation is exactly zero for $h = 3/2$. **N. B.** No other scaling solutions as the integrand is sign definite $\forall \alpha = 2h - 2$.

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- CFR scaling exponent corresponds to a constant flux of mass through the inertial range.
- More generally, we might have a mass dependent kernel and mass dependent diffusion coefficient:

$$\lambda \to \lambda(m_1, m_2),\ D \to D(m),$$

where λ is homogeneous of degree ζ. Then CFR becomes

$$\gamma_2 = 3 + \zeta.$$

Essential Ingredients for CFR

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- Nonlinear interactions which redistribute some conserved quantity between the modes of the system.
- Existence of an inertial range and a stationary constant flux state at large time.
- Identification of the correlation function responsible for transfer of flux.
- Scale invariance.

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These features are not unique to mass aggregation model. There should be a CFR in many (most) turbulence-like systems.

Energy CFR in wave turbulence: general considerations.

- The Hamiltonian is $H = \int d\vec{k} h(\vec{k})$,

$$h(\vec{k}) = \omega(\vec{k})\bar{a}(\vec{k})a(\vec{k}) + u(\vec{k})$$

where $u(\vec{k})$ is a non-linear part of Hamiltonian density. Let $U = \int d\vec{k} u(\vec{k})$.

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- Variables $\bar{a}(\vec{k}), a(\vec{k})$ are canonical, i. e.

$$\{\bar{a}(\vec{k}), a(\vec{k}')\} = i\delta^{(d)}(\vec{k} - \vec{k}'),$$

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- Continuity equation associated with conservation of energy:

$$\langle \dot{u}(\vec{k}) - \dot{\bar{a}}(\vec{k}) \frac{\delta U}{\delta \bar{a}(\vec{k})} - \dot{a}(\vec{k}) \frac{\delta U}{\delta a(\vec{k})} \rangle = 0.$$

Energy CFR in 3-wave turbulence.

- $u(\vec{k}_0) =$
  $\int\int d\vec{k}_1 d\vec{k}_2 \lambda(k_0; k_1, k_2)\delta(\vec{k}_{0,12}) \left(\bar{a}(\vec{k}_0)a(\vec{k}_1)a(\vec{k}_2) + c.\ c.\right)$,
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  where λ is homogeneous of degree γ.
- Continuity equation for energy density takes the form
  $$\int\int k_1^{d-1}dk_1 k_2^{d-1}dk_2\Big(\lambda(k;k_1,k_2)C(k,k_1,k_2)$$
  $$-\lambda(k_1;k,k_2)C(k_1,k,k_2)\Big) = 0,$$
  where $C(\vec{k}_1, \vec{k}, \vec{k}_2)$ is angle average of $\langle Re\left(a(\vec{k}_1)\dot{\bar{a}}(\vec{k})\bar{a}(\vec{k}_2)\right)\rangle$.
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  We assume that C scales with exponent h, to be determined.
- Zakharov transformation of the first integral:
  $k_1 = \frac{k}{k_1'}k,\ \ k_2 = \frac{k}{k_1'}k_2'.$

Energy CFR in 3-wave turbulence (continued).

- The result of applying ZT to the continuity equation is

$$\int\int k_1^{d-1} dk_1 k_2^{d-1} dk_2 \left(\left(\frac{k}{k_1} \right)^{3d+h+\gamma} - 1 \right) \lambda(k_1; k, k_2) C(k_1, k, k_2) =$$

which is identically satisfied if $h = -3d - \gamma$.

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- We conclude that $C\sim k^{-3d-\gamma}$.
- In the limit of weak non-linearity C can be expressed in terms of particle density $n(k)$ using weak turbulence closure:

$$C=Re\langle a_1\dot{\bar{a}}\bar{a}_2\rangle\sim\lambda n^2\omega\delta(\Delta\vec{k})\delta(\Delta\omega(k))\sim(k^{-d-\gamma})^2k^{-d+\gamma}\sim k^{-3d-\gamma},$$

which is consistent with CFR. However we expect CFR to hold even in the regime where weak turbulence approximation is not valid.

Energy CFR in 4-wave turbulence.

- $u(\vec{k}) = \frac{1}{2}\int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \, \lambda(k; k_1, k_2, k_3)\delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3)$
  $\left(a(\vec{k})\, a(\vec{k}_1)\, \bar{a}(\vec{k}_2)\, \bar{a}(\vec{k}_3) + c.\ c.\right)$

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- Continuity equation after ZT:
  $0 = \int (k_1k_2k_3)^{d-1} dk_1dk_2dk_3 \lambda(k,k_1,k_2,k_3) C(k,k_1,k_2,k_3)$ $\left[\left(\frac{k}{k_1}\right)^y + \left(\frac{k}{k_2}\right)^y + \left(\frac{k}{k_3}\right)^y - 3\right],$
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$Re\,\langle\dot{a}(\vec{k})\,a(\vec{k}_1)\,\bar{a}(\vec{k}_2)\,\bar{a}(\vec{k}_2)\rangle$; $y = 4d+\gamma+h$. h is a scaling exponent of C.
- Energy CFR: $h = -4d-\gamma$.
- Weak turbulence limit:
$C \sim \lambda n^3\omega\delta(\omega)\delta(k) \sim k^{\gamma-d}(k^{-2/3\gamma-d})^3 = k^{-\gamma-4d}$.

Wave action CFR in 4-wave turbulence (inverse cascade).

- Additional integral of motion: $N = \int d\vec{k}\bar{a}(\vec{k})a(\vec{k})$.

Wave action CFR in 4-wave turbulence (inverse cascade).

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$k_1^y \, \left[k_1^{-y} + k_2^{-y} - k_3^{-y} - k_4^{-y} \right],$
where $y = h + \gamma + 4d$ and C is angle average of
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- Weak turbulence limit:
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- Charge model has two integrals of motion: charge Q and $\langle Q^2 \rangle$. Q-flux is zeros but the flux of Q^2 is constant. Charge model provides an example of CFR associated with a quantity conserved only on average.

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Constant flux state is local if

$$\sigma > \frac{1}{2}(\nu - \mu - 1)$$

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- **Open question** : what happens for systems where locality is not expected to hold?