Spatial dependence of passive scalar correlation functions in decaying problem in random smooth velocity field

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Equation of passive scalar evolution

$$\partial_t \theta + (\mathbf{v} \cdot \nabla)\theta = \kappa \Delta \theta, \qquad \theta(\mathbf{r}, \mathrm{t}=0) = \theta_0(\mathbf{r})$$

velocity of fluid v does not depend on θ

Correlation length of scalar is much smaller than correlation length of velocity

Model

Origin of the reference frame moves with some Lagrangian trajectory $\mathrm{r}_0(t)$

Profile of velocity of fluid is linear at scales of interest. In the reference frame

$$\mathbf{v}(\mathbf{r}) = \mathcal{V}(\mathbf{r}_0(\mathrm{t})) - \mathcal{V}(\mathbf{r}_0(\mathbf{t}) + \mathbf{r}) \approx \hat{\sigma}\,\mathbf{r}$$

so-called Batchelor regime

$\hat{\sigma}(t)$ -- matrix of velocity strain; its statistics is isotropic

$\mathcal{V}$ -- velocity of fluid in fixed reference frame

$$\partial_t \theta + (\mathbf{r} \cdot \hat{\sigma}^T \cdot \nabla)\theta = \kappa \Delta \theta$$

Correlation function of passive scalar

$$\mathcal{F}_n(\{\mathbf{r}_\mathrm{i}\}, \mathrm{t}) = \left\langle \prod_{i=1}^{n} \theta(\mathbf{r}_\mathrm{i}, t) \right\rangle$$

Correlation function depends on only mutual displacement of points $\mathbf{r}_i$

In our model

averaging over

a) statistics of initial passive scalar distribution $\theta_0(\mathbf{r})$

b) over statistics of random process $\hat{\sigma}(t)$

M. Chertkov and V. Lebedev, Phys.Rev.Let., **90**, 034501 (2003)

E. Balkovsky and A. Fouxon, Phys.Rev.E, **60,** 4164 (1999)

Physical systems

1. Kolmogorov turbulence

Scales:

L_{int} -- integral scale, scale of energy pumping

η -- viscous (Kolmogorov) scale

Scales of interest:

scalar mixing at subviscous scales << η

At the scales velocity profile is linear

Averaging over volume at fixed time *t*

$$\mathcal{F}_n = \frac{1}{\mathcal{V}} \int \prod_i \theta(\mathbf{r}_i + \mathbf{r}'; t)\, \mathrm{d}\mathbf{r}'$$

$\mathcal{V}$ **-- volume of`the system**

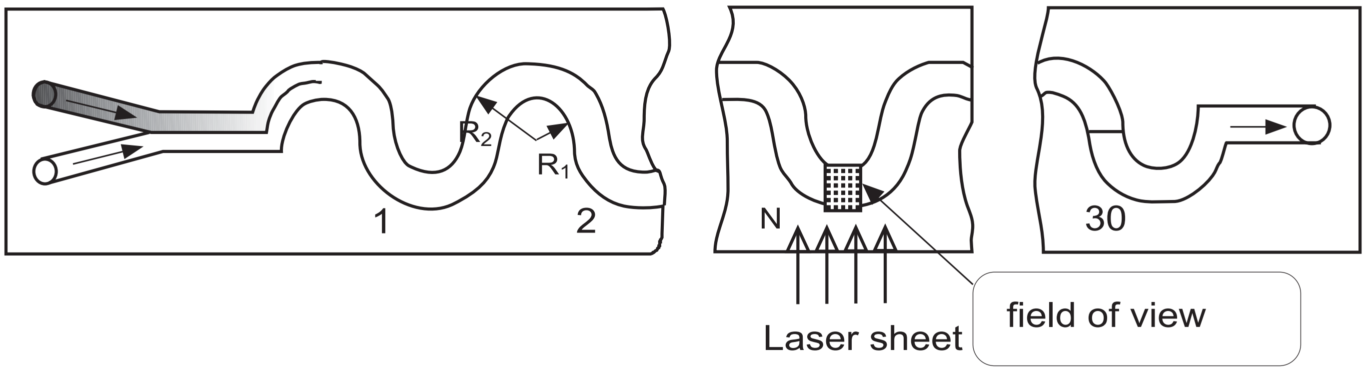
Physical systems

2. Elastic turbulence

A.Groisman, V.Steinberg, Nature 410, 905 (2001)

Weak polymer solution

Chaotic flow in curved pipe
linear velocity profile at scales << L



Reynolds number Re << 1

Weissenberg number $\mathrm{Wi} = \mu \frac{\mathrm{V}}{\mathrm{L}} \sim 1$

L – diameter of the pipe
V – velocity of fluid
μ **-- relaxation time of polymer**

Time if mixing is determined by length *s* along the pipe and mean velocity: $t = s / \langle \mathrm{v} \rangle$

Averaging over a) time
b) over cross area (optional)

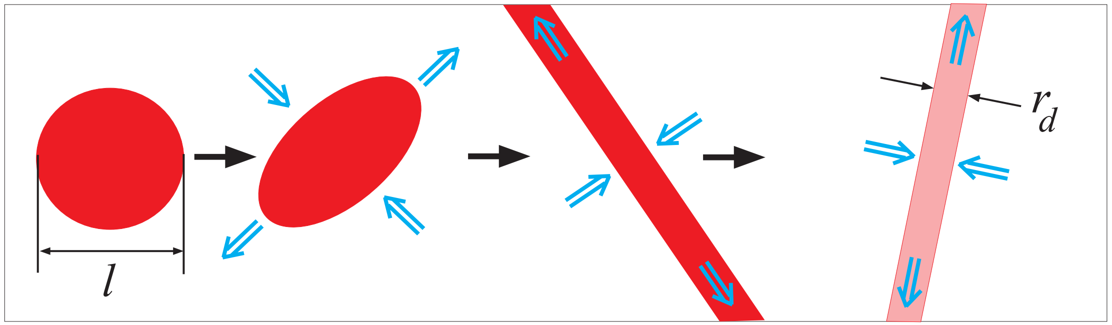
Evolution of a blob

$$\partial_t \theta + (\mathbf{r} \cdot \hat{\sigma}^T \cdot \nabla)\theta = \kappa \Delta \theta$$

$$\theta_0(\mathbf{r}) = \Theta_0(r/l)$$

decreases at distances r ~ l

$$\mathrm{Pe} = \frac{l}{\mathrm{r_d}} >> 1$$



$$\theta(\mathbf{r},t) = \Theta(\mathbf{r},t)$$

Blob has dimensions

$$l_1 > \ldots > l_d$$

Smallest dimension reaches $\mathrm{r_d}$ **at time**

$$t_d \sim \lambda^{-1} \ln \mathrm{Pe} >> \lambda^{-1}$$

Lagrangian trajectory

$$\mathbf{r}(t) = \hat{W}(t)\,\mathbf{r}(0),$$

$$\frac{\mathrm{d}\hat{W}}{\mathrm{d}t} = \hat{\sigma}\hat{W}$$

Lyapunov exponent

$$\lambda = \left\langle \frac{\mathrm{d}\ln \mathrm{r}}{\mathrm{d}t} \right\rangle_\sigma$$

At typical realization of velocity $\hat{\mathrm{W}}(\mathrm{t}) \propto e^{\lambda t}$

Diffusion scale

$$\mathrm{r_d} = \sqrt{\kappa/\lambda}$$

Kolmogorov turbulence $\mathrm{r_d} << \eta \Leftrightarrow \kappa << \nu$

Elastic turbulence $\mathrm{r_d} << L \Leftrightarrow \kappa << \nu\,\mathrm{Re}$

$$\mathrm{Pr} = \frac{\nu}{\kappa} >> 1$$

ν -- coefficient of kinematic viscosity

Statistically homogeneous distribution

Many chaotically scattered blobs

Mean over space $\langle \theta_0 \rangle = 0$

$$\theta_0 = \sum_i a_i \Theta_0(| \quad - \quad | / l)$$

Mean distance between blobs D << l

Gaussian statistics with zero mean

Pair correlation function: averaging over blob positions $\mathbf{r}_i$

$$\mathcal{F}_2(\mathbf{R}, t=0) \equiv \langle \theta_0(\quad + \quad)\theta_0(\quad) \rangle =$$

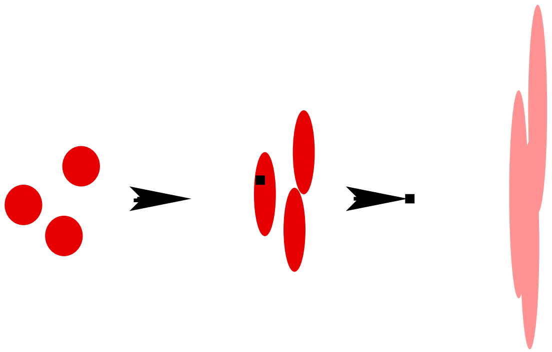
$$= \left(\frac{l}{D}\right)^d \langle a^2 \rangle \int \frac{\mathrm{d}^d r'}{l^d} \Theta_0(|\mathbf{r}' + \mathbf{R}/2|/l)\Theta_0(|\mathbf{r}' - \mathbf{R}/2|/l)$$

Two points of correlation function, $\quad + \quad$ must fall into the same blob

Pair correlation function

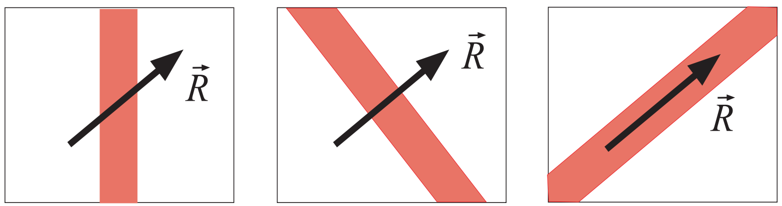
$$\mathcal{F}_2(\mathbf{R},t) \equiv \langle \theta(\quad + \quad ,t)\theta(\quad ,t)\rangle =$$

$$= \langle a^2 \rangle \left\langle \frac{l_1 \ldots l_d}{D^d} \int \frac{\mathrm{d}^d r'}{l_1 \ldots l_d} \Theta(|\mathbf{r}' + \mathbf{R}/2|;t)\ \Theta(|\mathbf{r}' - \mathbf{R}/2|;t) \right\rangle_\sigma$$



Blob dimensions: $l_1 > \ldots > l_d$

Averaging over rotating:

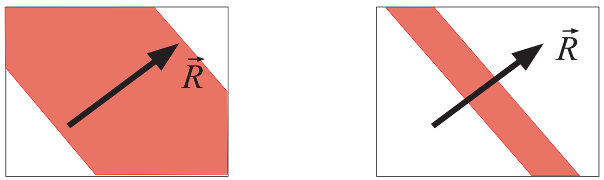


2-dimensional case:

$\mathrm{R} < l_2, \quad \sim 1$

$\mathrm{R} >> l_2, \quad \frac{l_2}{\mathrm{R}}$

Averaging over degree of elongation:
requires knowing of statistics of blob dimensions $l_1, \ldots, l_d$



One-point moments: $\langle |\theta|^\alpha \rangle$ E. Balkovsky and A. Fouxon, Phys.Rev.E, 60, 4164 (1999)

Statistics of blob dimensions. Entropy (Kramer) function S.

Lagrangian trajectory

$$\mathbf{r}(t) = \hat{W}(t)\,\mathbf{r}(0) \qquad \frac{d\hat{W}}{dt} = \hat{\sigma}\hat{W}$$

$$\hat{W}(t) = \hat{N}\begin{pmatrix} e^{\rho_1} & 0 & 0 \\ 0 & e^{\rho_2} & 0 \\ 0 & 0 & e^{\rho_3} \end{pmatrix}\hat{O}$$

$\hat{N}, \hat{O}$ -- orthogonal matrices

Incompressibility of flow

$$\rho_1 > \rho_2 > \rho_3, \qquad \rho_1 + \rho_2 + \rho_3 = 0$$

Lyapunov exponents:

$$\lambda_i = \langle \dot{\rho}_i \rangle = \langle \tilde{\sigma}_{ii} \rangle, \qquad \tilde{\sigma} = \hat{N}^T \sigma \hat{N}$$

Main Lyapunov exponent $\lambda = \lambda_1$

Joint probability distribution function at times $t \gg 1/\lambda$, when $e^{\rho_1} \gg e^{\rho_2} \gg e^{\rho_3}$

$$\mathcal{P}(\rho) \propto e^{-tS\left(\frac{\rho_2}{t},\frac{\rho_3}{t}\right)}\delta(\rho_1 + \rho_2 + \rho_3)$$

Entropy function S is convex and has minimum at $\rho_i = \lambda_i t$

Solution in Fourier-space of equation $\partial_t \theta + (\mathbf{r}\cdot\hat{\sigma}^T\cdot\nabla)\theta = \kappa\Delta\theta$

$$\theta(\mathbf{k},t) = \theta_0(\hat{W}^T\mathbf{k},t)\ \exp\left[-r_d^2\mathbf{k}\hat{W}\hat{\Lambda}\hat{W}^T\mathbf{k}\right], \qquad \hat{\Lambda} = \int_0^t \lambda\, dt'\, \hat{W}^{-1}(t')\hat{W}^{-1,T}(t')$$

Results, 3D-case

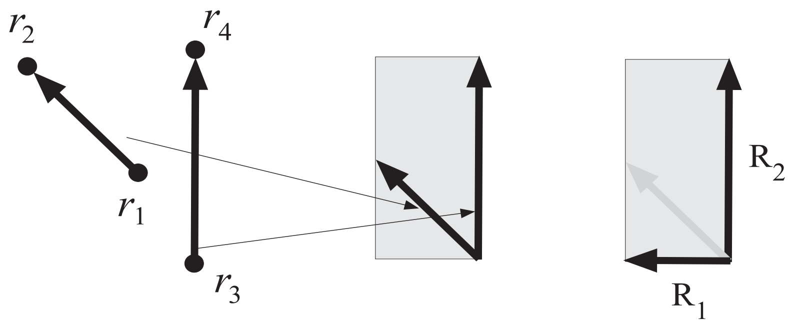
Pair correlation function

$$\mathcal{F}_2(\mathrm{R};t) = e^{-\gamma_2 t}\left(\frac{R}{l}\right)^{a_2}, \qquad 1 < \mathrm{a}_2 < 2, \quad \mathrm{a}_2 = \mathrm{a}_2(\mathrm{S})$$

Fourth order correlation function

$$\mathcal{F}_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4; t) =$$

$$= \sum_{i=1}^{3} F_4(\mathbf{R}_{i,1}, \mathbf{R}_{i,2}; t)$$



$$F_4 = C_2^2 \mathrm{Pe}^{\,\mathrm{x}} e^{-\gamma_4 t}\left(\frac{\mathrm{r_d}}{\mathrm{R}_1}\right)^{a_4}\left(\frac{\mathrm{r_d}}{\mathrm{R}_2}\right)^{b_4},$$

$1 \le a_4 \le 2, \quad b_4 = 1$

$a_4 = 2, \quad 1 \le b_4$

$\mathrm{x} = a_4 + b_4, b_4 < 2, \quad \mathrm{x} = 2, \quad b_4 > 2$

$$\mathcal{F}_4 = C_2^2 e^{-\gamma_4 t} \mathrm{Pe}^{\,x}\left(\frac{r_d}{r_{12}}\right)^{a_4}\left(2\left(\frac{r_d}{r_{23}}\right)^{b_4} + 1\right) \quad \rightarrow \quad \mathcal{F}_4 = C_2^2 e^{-\gamma_4 t} \mathrm{Pe}^{\,x}\left(\frac{r_d}{r_{12}}\right)^{a_4}, \qquad a_4 < 2a_2$$

Summary

1) It is possible to bind passive scalar correlation functions with Kramer function at arbitrary velocity statistics
2) Passive scalar correlation functions decays as powers of distances between points in correlation function. The fact represents foliated structure of passive scalar distribution in space
3) Statistics of passive scalar is strongly nongaussian and intermittent.

Pair correlation function

One-point moment

$$\langle \theta^2 \rangle = C_2 \left\langle \frac{\mathrm{V}}{l^3} \Theta^2 (r=0) \right\rangle$$

Volume of blob $\mathrm{V} = l_1 l_2 l_3$

Typical value of scalar in blob $\Theta(r=0) = \frac{l^3}{V}\Theta_0(r=0)$

$$C_2 = \left(\frac{l}{D}\right)^3 \langle a^2 \rangle$$

$$\langle \theta^2 \rangle = C_2 l^3 \left\langle \frac{1}{\mathrm{V}} \right\rangle = C_2 \int \mathrm{d}\rho_2 \mathrm{d}\rho_3\, e^{-tS - \sum_{i=1,2} \chi(-\rho_i - \ln \mathrm{Pe})} = C_2 \mathrm{Pe}^{x} e^{-\gamma_2 t}$$

If at point $\rho_2 = 0$, $\partial_3 S = 1$

$\partial_2 S < 0,$	then	$\rho_2^* = \lambda_2^* t, \quad \lambda_2^* > 0$	$a = 1$
$1 < \partial_2 S < 0,$	then	$\rho_2^* = -\ln \mathrm{Pe}$	$a = 1 + \partial_2 S$
$\partial_2 S > 1,$	then	$\rho_2^* = \lambda_2^* t, \quad \lambda_2^* < 0$	$a = 2$

$$l_i(\rho_i) = l \exp[-\ln \mathrm{Pe} + \chi(\rho_i + \ln \mathrm{Pe})]$$