Role of elastic stresses in statistical and structural properties of elastic turbulence and polymer stretching

V. Steinberg

in collaboration with
T. Burghelea, E. Segre, S. Gerashchenko, C. Chevallard

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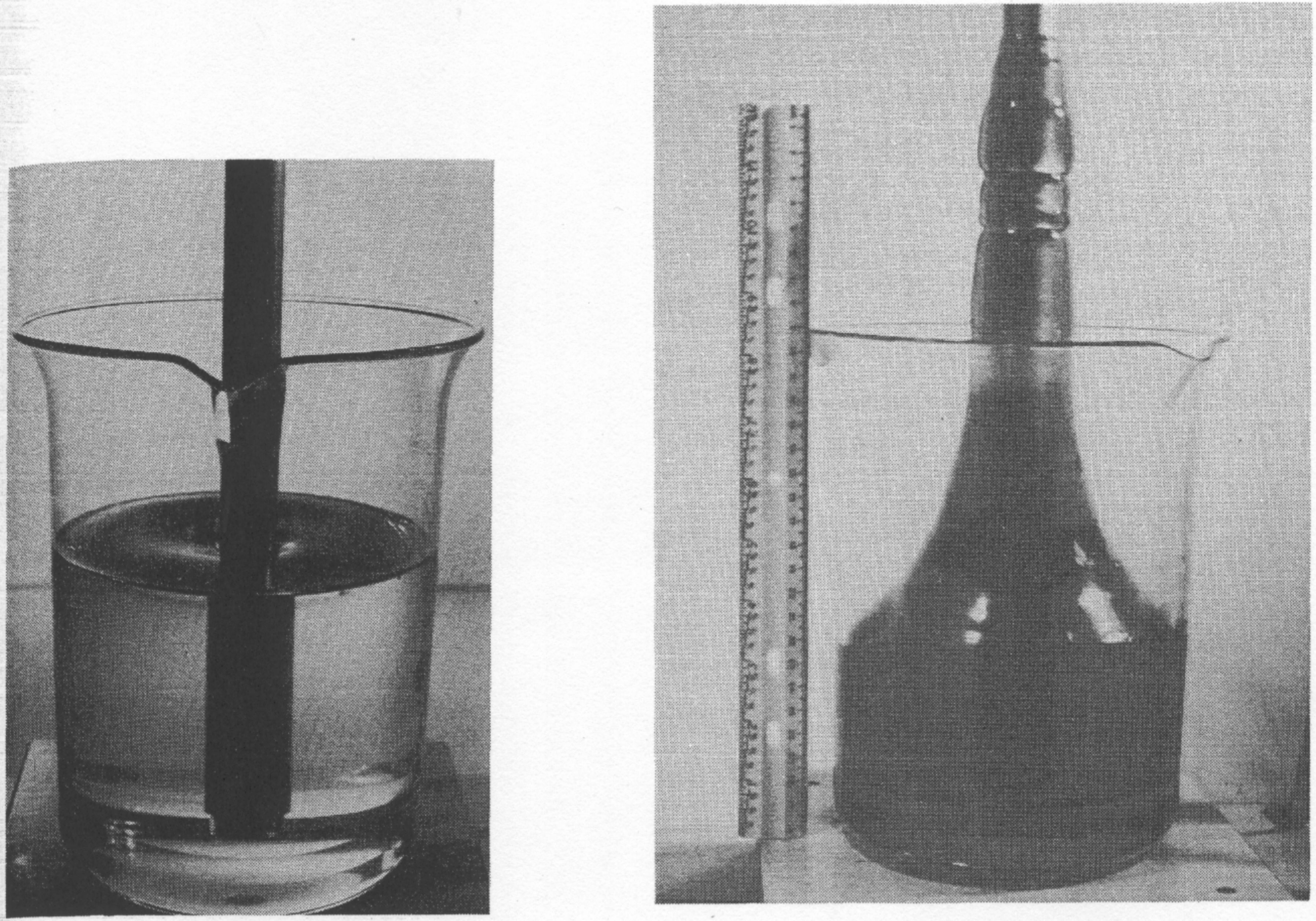
Two aspects of polymer hydrodynamics:

1. *Influence of polymers on flow stability, flow structure and statistics- macro-hydrodynamic experiment*

2. *Influence of flow on polymer dynamics, conformations and statistics- micro-hydrodynamic experiment, dynamics of a single molecule*

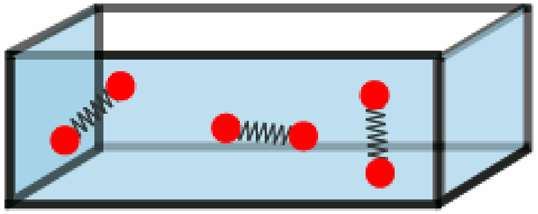


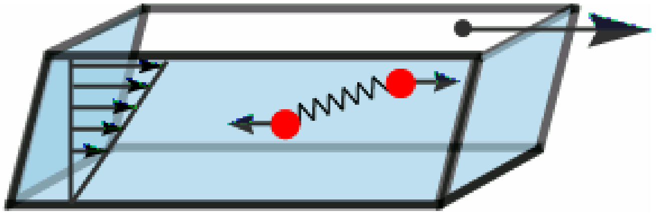
N

Rod climbing, or Weissenberg effect

Polymer molecules in a shear flow.

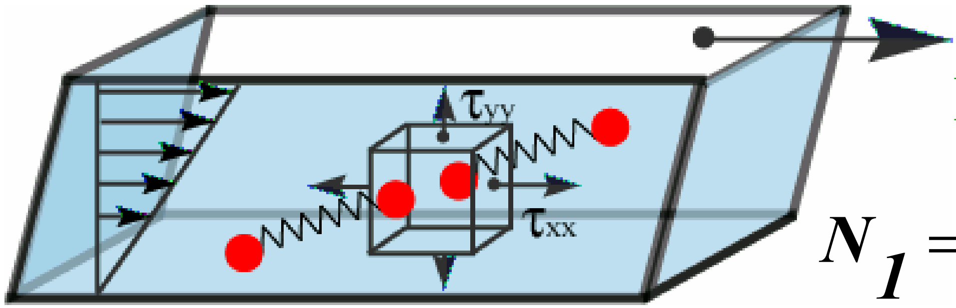
Elastic dumbbell model:





$\lambda \propto \eta R/k$ **-polymer relaxation time**

η-viscosity, R-molecule size; k~1/R²-spring constant.



First normal stress difference:

$$N_1 = \tau_{xx} - \tau_{yy} \propto (\lambda\dot{\gamma})^2, \dot{\gamma} \text{ - shear rate.}$$

Hydrodynamics of polymer solutions

Equation of motion:

$$\frac{\partial \vec{\mathbf{V}}}{\partial \mathbf{t}} + \underbrace{(\vec{\mathbf{V}}\,\vec{\nabla})\vec{\mathbf{V}}}_{\text{Inertial nonlinearity}} = -\vec{\nabla}\mathbf{p}/\rho - \underbrace{\vec{\nabla}\tilde{\boldsymbol{\tau}}/\rho}_{\text{linear}}$$

Constitutive equation for Oldroyd-B model

- Full stress tensor- $$\tilde{\tau} = \tilde{\tau}_S + \tilde{\tau}_P$$

- For solvent part- $$\tilde{\tau}_S = -\eta_S\left[\vec{\nabla}\vec{\mathbf{V}} + (\vec{\nabla}\vec{\mathbf{V}})^T\right]$$

- For polymer part $$\underbrace{\tilde{\tau}_P}_{\text{Linear relax}} + \lambda\frac{\mathbf{D}\tilde{\tau}_P}{\mathbf{Dt}} = -\eta_P\left[\vec{\nabla}\vec{\mathbf{V}} + (\vec{\nabla}\vec{\mathbf{V}})^T\right]$$

and convective derivative:

$$\frac{D\tilde{\boldsymbol{\tau}}_P}{Dt} \equiv \frac{\partial\tilde{\boldsymbol{\tau}}_P}{\partial t} + \underbrace{(\vec{V}\,\vec{\nabla}\,)\tilde{\boldsymbol{\tau}}_P - \tilde{\boldsymbol{\tau}}_P(\vec{\nabla}\vec{V}) - (\vec{\nabla}\vec{V})^T\,\tilde{\boldsymbol{\tau}}_P}_{\text{elastic nonlinearity}}$$

$$\mathrm{Wi} = \mathrm{V}\lambda/\mathrm{L} = \frac{\text{nonlinearity}}{\text{relaxation}}$$

Weissenberg number

$$\mathrm{Re} = VL/\nu = \frac{\text{nonlinearity}}{\text{dissipation}}$$

Reynolds number

Three limiting cases:

$$1.\ \mathrm{Re} \gg 1; Wi \Rightarrow 0$$ Hydrodynamic turbulence

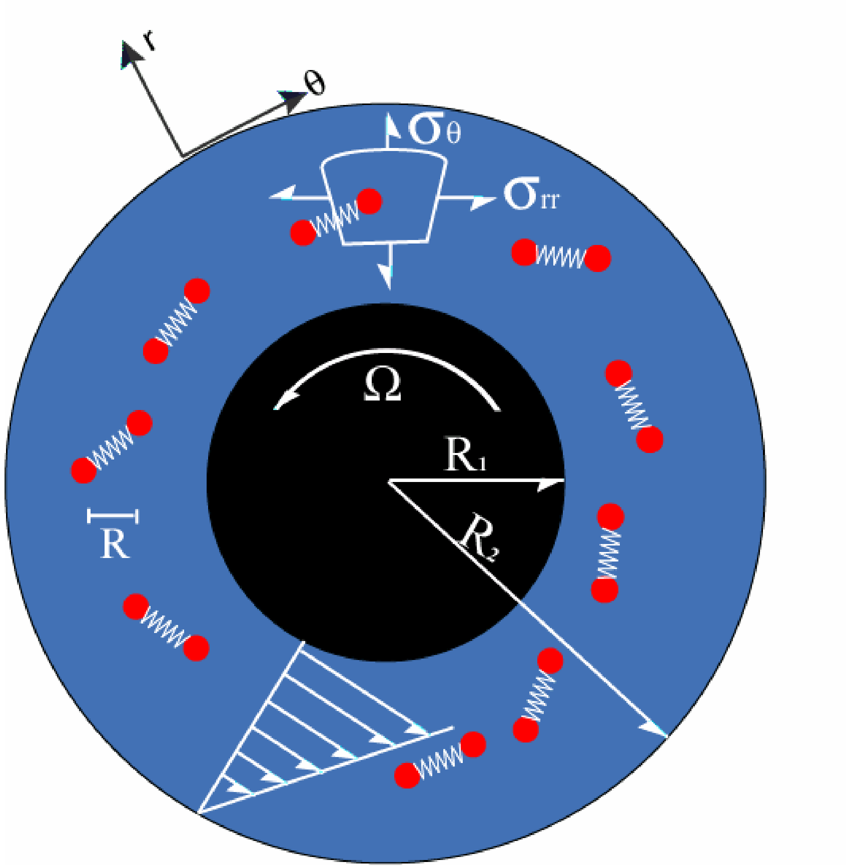
$$2.\ Wi \gg 1; \mathrm{Re} \Rightarrow 0$$ Elastic turbulence

$$3.\ \mathrm{Re} \gg 1; Wi \gg 1$$ Turbulent drag reduction

Elastic stress is the main source of nonlinearity at large Wi and low Re.

Recently discovered Elastic turbulence is a dynamic state solely driven by the nonlinear elastic stresses that are present in the flow of a dilute polymer solution. It can be excited at arbitrarily small Re (being independent on it) but only, if Wi exceeds a critical value (usually of order of unity).

Rod Climbing (Weissenberg) Effect.



Volume force: $\frac{N_1}{r}$

(hoop stress).

$$N_1 = \sigma_{\theta\theta} - \sigma_{rr} \propto \langle R^2 \rangle Wi^2$$

$$Wi = \lambda \dot{\gamma}, d = R_2 - R_1$$

$$\dot{\gamma} = \Omega \frac{R_1}{d}$$ **-shear rate.**

Criterion of elastic instability in the framework of Oldroyd-B model

(Larson, Shaqfeh, Muller,1990).

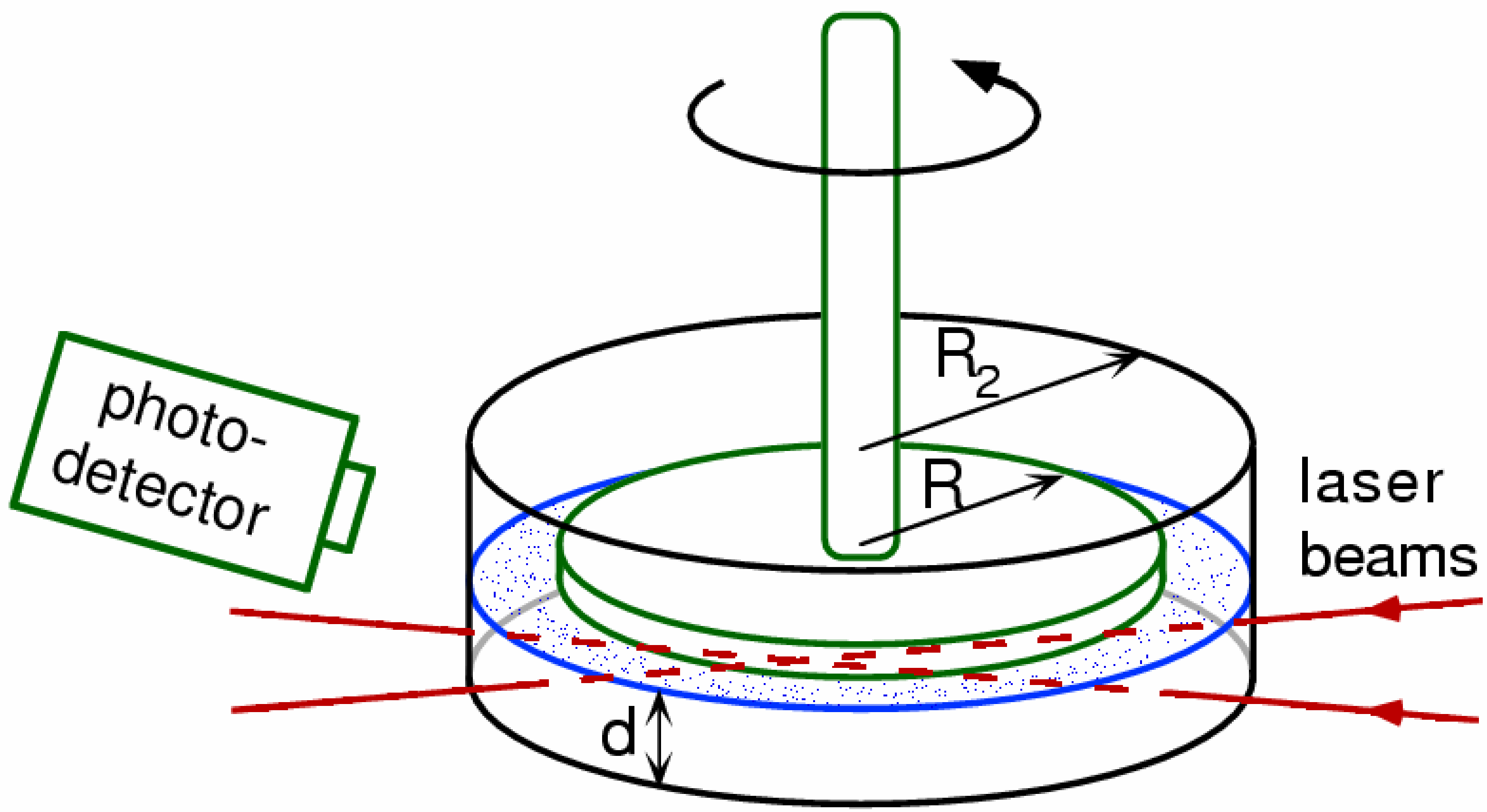
$$K \equiv \frac{\eta_P}{\eta}\frac{d}{R}Wi^2 = const$$

Strong shear primary flow → Polymer stretching

Weak radial elongation coupled to strong primary shear flow leads to energy release into secondary flow due to increase in hoop stress

Elastic turbulence

A. Groisman & V.Steinberg, ***Nature* 405**, 53 (2000)

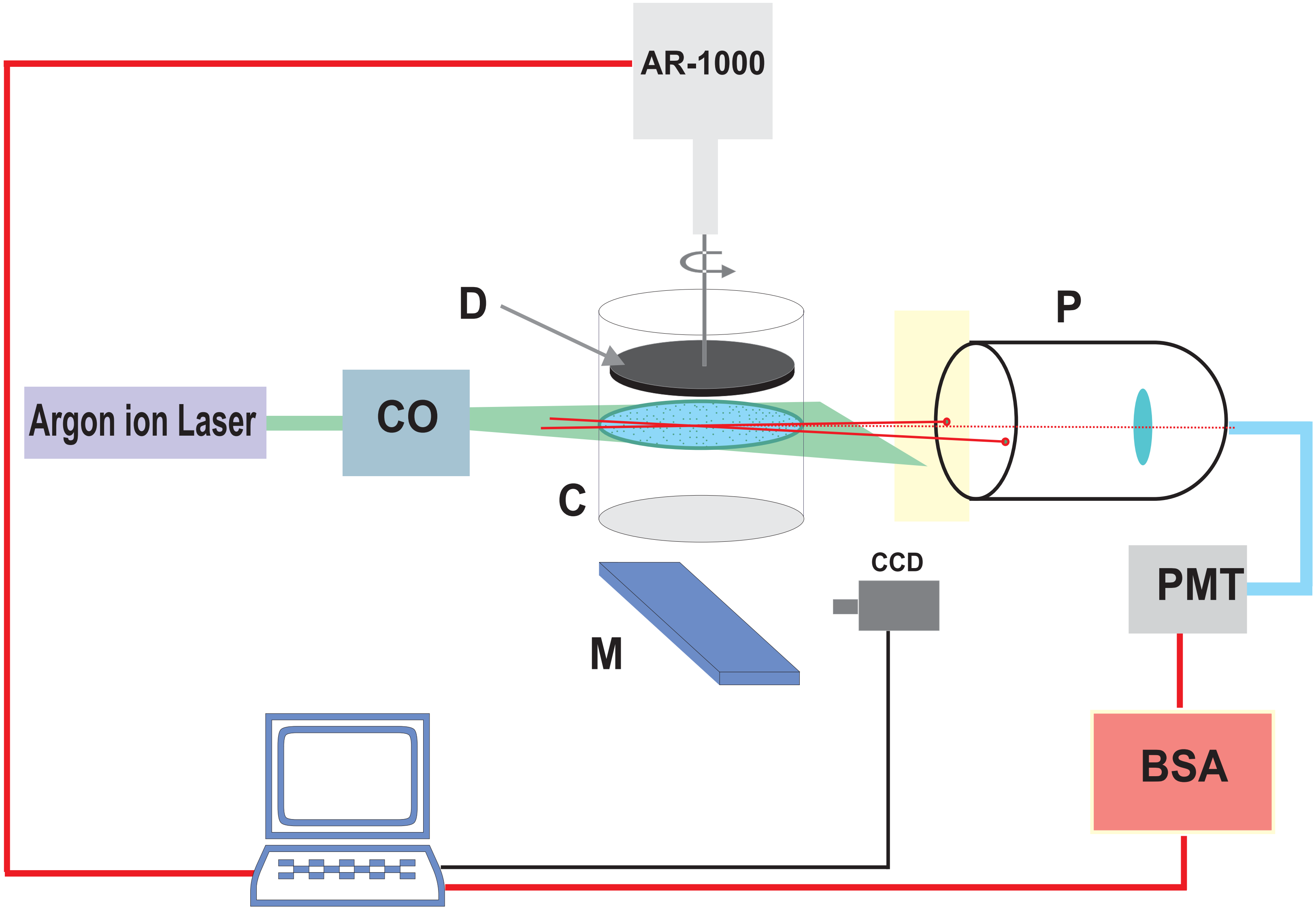


- **Solution:** 80 ppm PAAm, 65% saccharose and 1% NaCl,

$$\eta_s = 0.324 Pas, M = 18 \cdot 10^6 da; \eta = 0.424 Pas \text{ at } \dot{\gamma} = 1 s^{-1}$$

relaxation time λ=3.4 sec at 12C (temperature at which the experiment run)

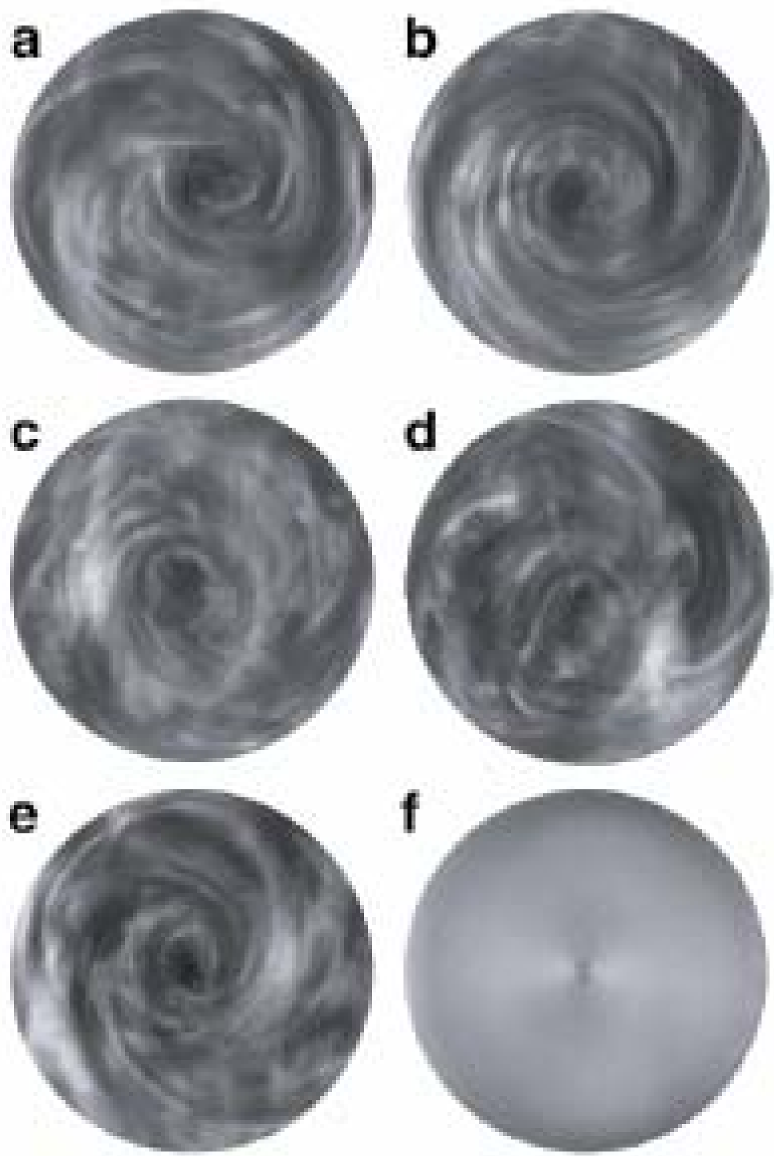
Von Karman Swirling Flow System



MAIN FEATURES OF ELASTIC TURBULENT FLOWS:

- randomness (in time), spatially smooth flow fields
- sharp growth of the flow resistance
- algebraic decay of power spectra
- efficient mixing

Visualization of Elastic Turbulence.



Wi=6.5; Re=0.35 (a,b)

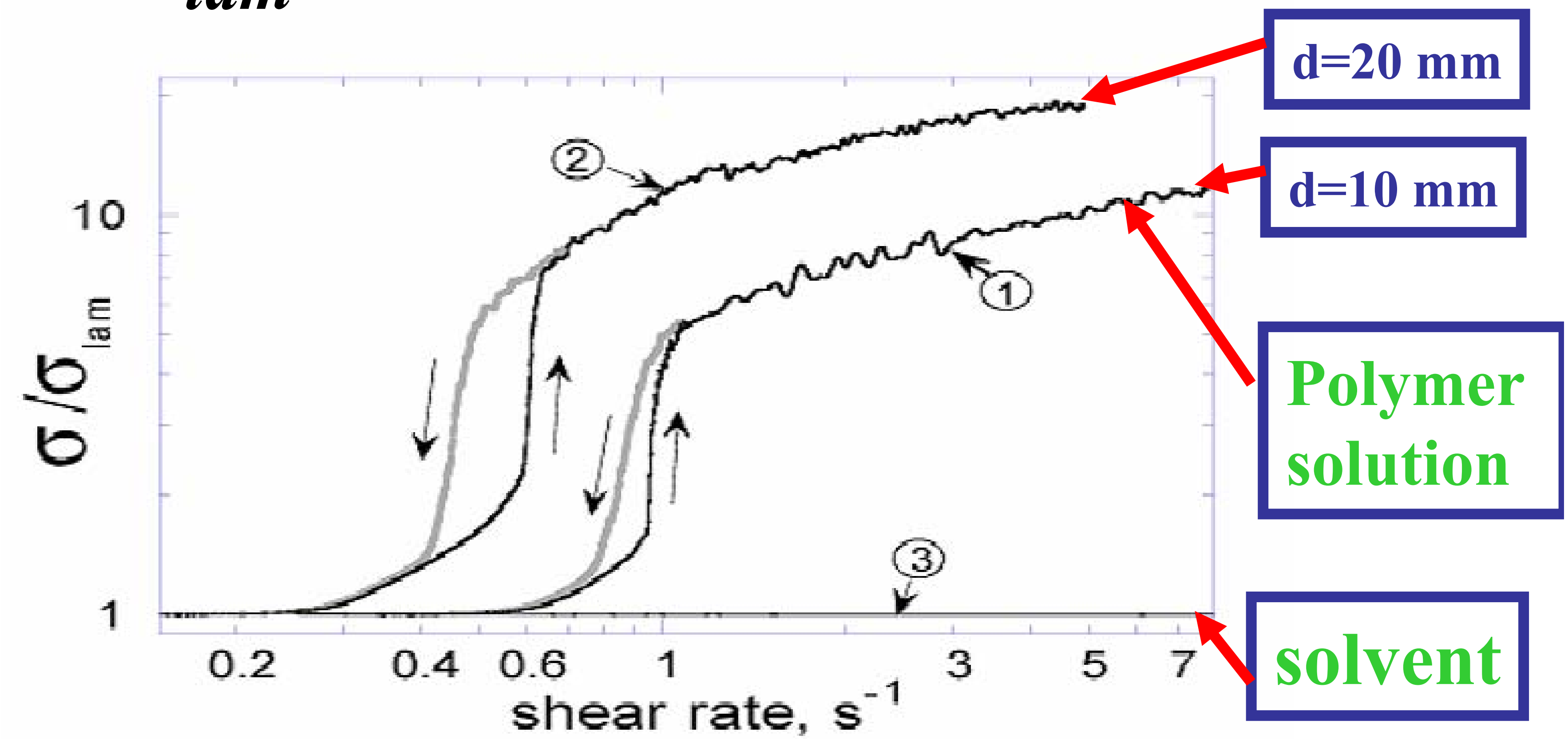
Wi=13, Re=0.7 (c-e)

Re=1, pure solvent

Flow Resistance

σ-stress measured at the upper plate

σ_{lam} -stress in laminar shear flow



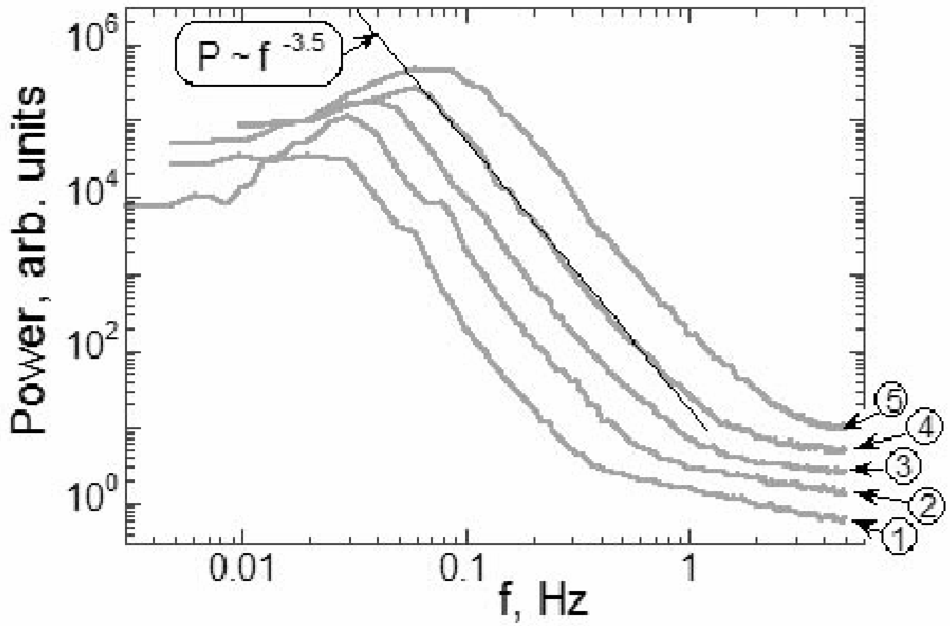
Shear rate 1 1/s--Wi=3.4, Re=0.3

Flow resistance increases by factors of:

11.5 for d=10 mm and 19 for d=20 mm

as in a pipe at

$\mathrm{Re} = 10^5$



Power spectra of azimuthal velocity fluctuations at (1) Wi=4.3; (2) Wi=6.3; (3) Wi=9.2; (4) Wi=13.6; (5) Wi=20.1

(LDV measurements)

Fluid motion is excited in a wide range of temporal scales, due to fast decay only low frequencies contribute to power

Elastic turbulence is "counterintuitive"

- Dependence on fluid viscosity, η

Inertial turbulence: higher $\eta \longrightarrow$ higher V,

Elastic turbulence: higher $\eta \longrightarrow$ lower V,

that is needed to excite turbulence

- Dependence on system size, d

Inertial turbulence: smaller d $\longrightarrow$ higher V,

Elastic turbulence: smaller d $\longrightarrow$ lower V,

that is needed to excite turbulence

- No apparent spatial scale other than d

only one time scale, λ , exists

Spatially smooth temporally random flow:

(i) in inertial turbulence at scales below dissipation length
(ii) in elastic turbulence

Properties of smooth random flow:

1. Velocity power spectrum decays faster than k^{-3}

2. Smallness of higher order space derivatives in velocity field or in cross-correlation function of velocity field. Thus,

$$V_i(r,t) = V_i(r(0),t) + \frac{\partial V_i}{\partial r_j}(r_j - r(0)) + \ldots$$

$$\frac{\partial V_i}{\partial r_j} >> \left(\frac{\partial V_i}{\partial r_j}\right)^2$$

$$\boxed{\textbf{Batchelor regime}}$$

It was shown experimentally by approximation a shape of velocity spatial correlation function just by velocity gradient terms (T. Burghelea, E. Segre & V. Steinberg, Phys. Fluids 17, 103101 (2005))

Theory of Elastic Turbulence

E. Balkovsky, A. Fouxon, V. Lebedev, ***PRE*** **64**, 056301 (2001)
A. Fouxon, V. Lebedev, ***Phys. Fluids*** **15**, 2060 (2003)

In the framework of molecular theory polymer stress tensor can be expressed as

$$\tau_{p,ij} \propto n\tau^{-1}\langle R_i R_j \rangle$$

*where **n** is polymer concentration, τ is relaxation time , and R_i end-to-end vector and $\langle R_i R_j \rangle$ the average conformation tensor*

In Hookean approximation of polymer elasticity and by neglecting thermal noise one gets uniaxial tensor $\longrightarrow$ $\tau_{ij} = B_i B_j$

Analogy with a small scale magneto-dynamo

Then equations of motion can be rewritten in the form similar to MHD equations with zero magnetic resistance and linear damping at Re<<1 and Wi>>1

$$\frac{\nabla P}{\rho} = \left(\vec{B}\nabla\right)\vec{B} + \nu\Delta\vec{V}; \nabla\vec{V} = 0 \qquad \textbf{(1)}$$

$$\partial_t\vec{B} + \left(\vec{V}\nabla\right)B = \left(\vec{B}\nabla\right)\vec{V} - \frac{\vec{B}}{\tau}; \nabla\vec{B} = 0 \qquad \textbf{(2)}$$

Eqs. (1,2) show instability at Wi>1 that leads to random statistically steady state. The latter is stabilized probably due to back reaction of stretched polymers on velocity field (Eq.(2)). In the case, when viscous and relaxation dissipations of the same order, one gets

$$\boxed{\left(\frac{\partial V_i}{\partial r_j}\right)_{rms} \propto \tau^{-1}}$$

Then elastic stress can be estimated as

$$\boxed{\tau_p = B^2 \propto \nu \nabla_i V_j \propto \frac{\eta}{\tau}} \longrightarrow$$

and saturates as well as rms of velocity gradients

Velocity Spectrum in Elastic Turbulence

Problem of small scale fluctuations ***v, B'*** based on Eqs. (1,2) is reduced to linearly decaying passive field problem advected in large scale fields ***V,B*** that leads to power law velocity spectrum (spherically normalized)

$$\mathrm{E}(k) \propto (kL)^{-\beta};(\beta > 3)$$

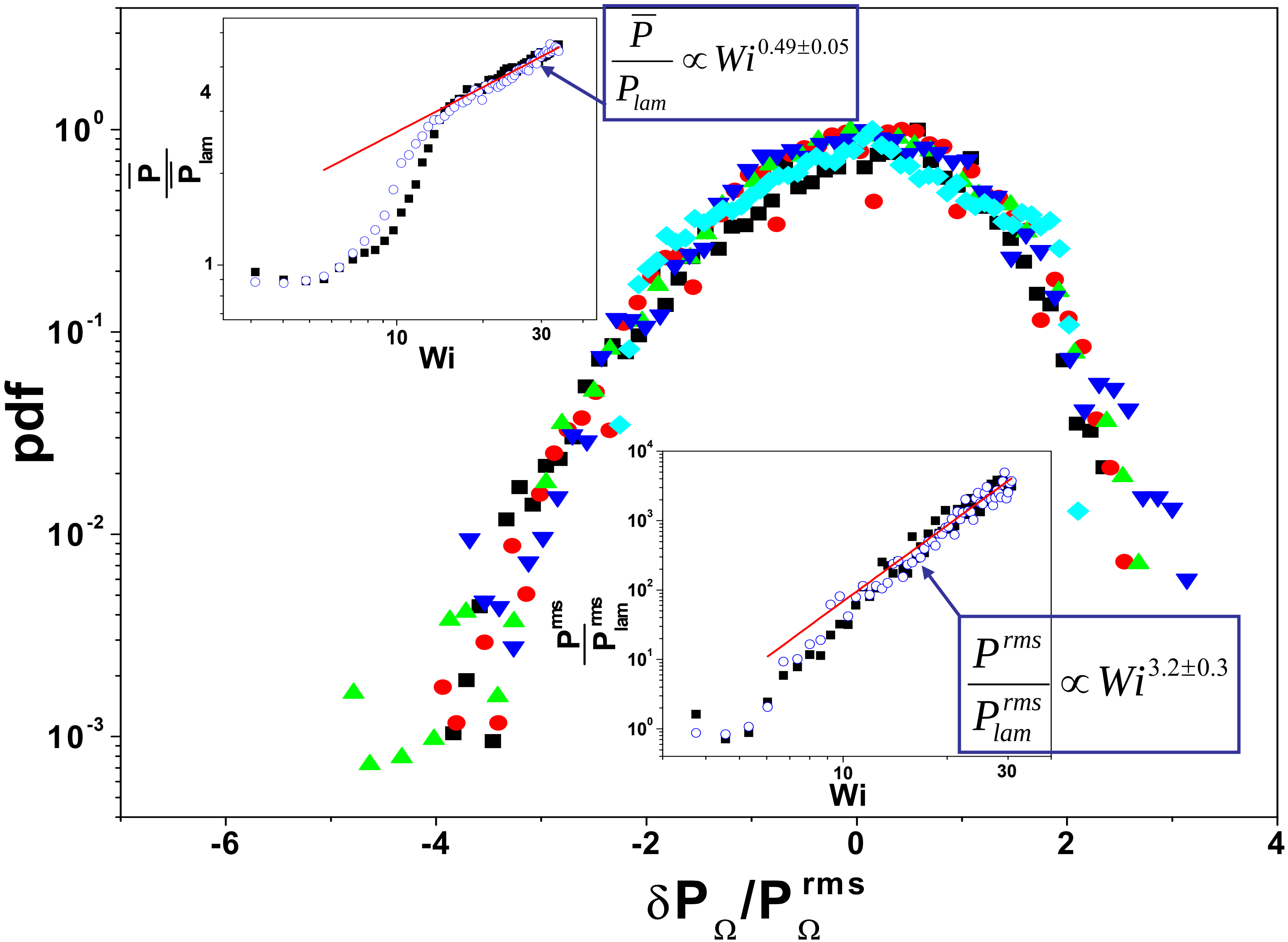
$$\beta = 3.3 \div 3.5$$ ***from our experiment***

Exploring analogy of elastic stress field with magneto-dynamo and passive scalar advection

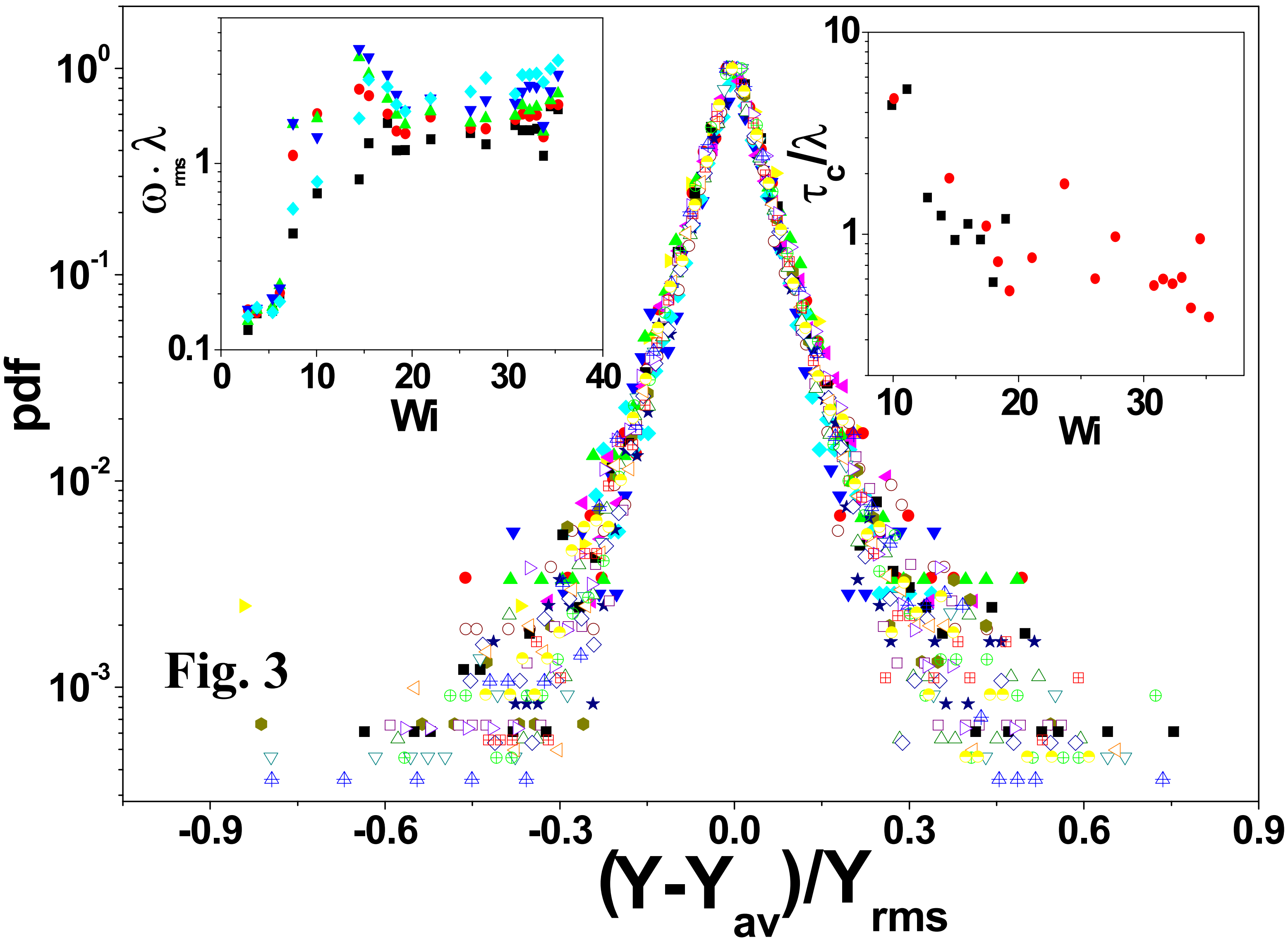
T. Burghelea, E. Segre, V. Steinberg, submitted to PRL (2005)

THE MAJOR STEPS IN UNDERSTANDING THE MECHANISM OF ELASTIC TURBULENCE:

- view the elastic turbulence as a turbulence of the field of elastic stresses, τ_p
- the stress field is directly related to the stretching of polymer molecules in the random flow: $\tau_p \propto \langle R_i R_j \rangle$ and is coupled to coil-stretched transition on polymer .
- large scale properties of τ_p can be inferred from measurements of global quantities, such as the power injected into the system
- local properties of the stress fields can be inferred from local measurements of fluctuations of the velocity gradients (vorticity)

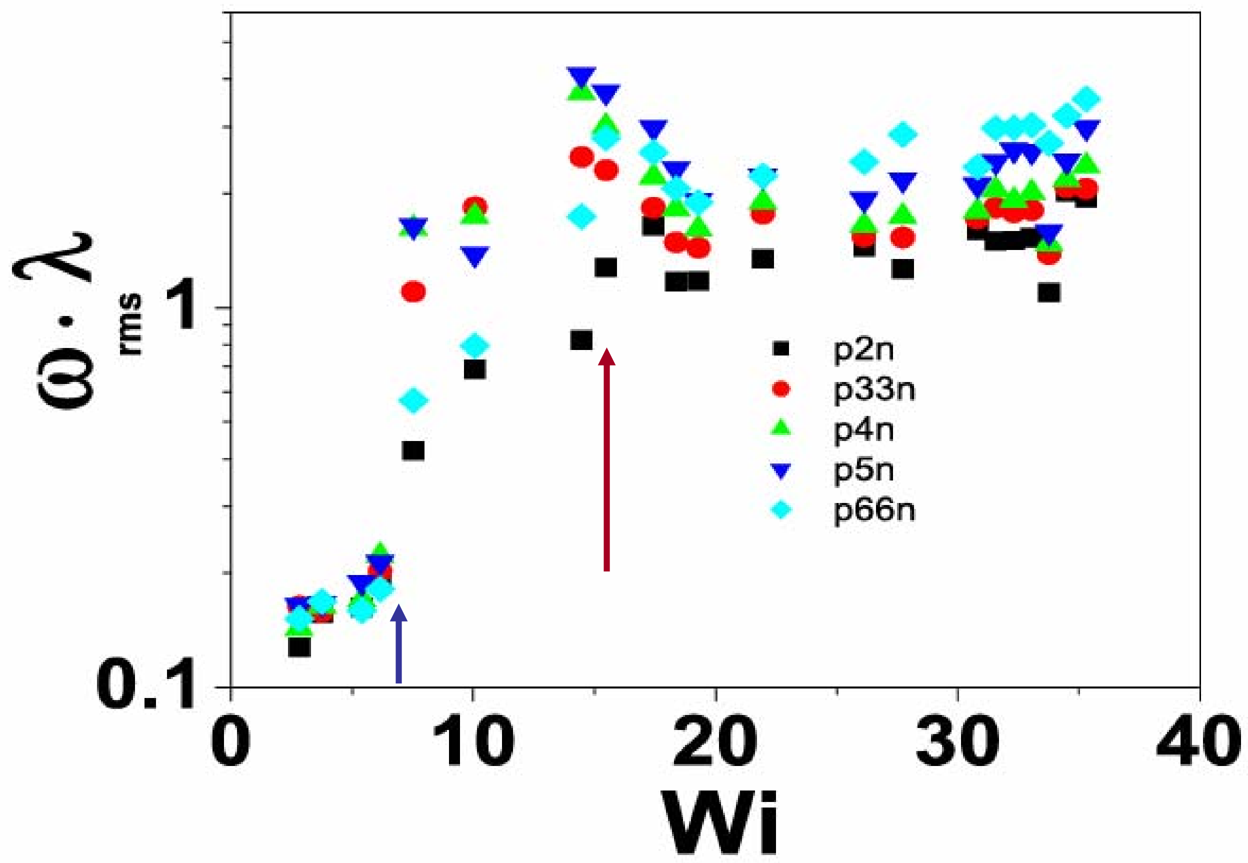


Probability distribution function of injected power fluctuations at different Wi in the elastic turbulence regime

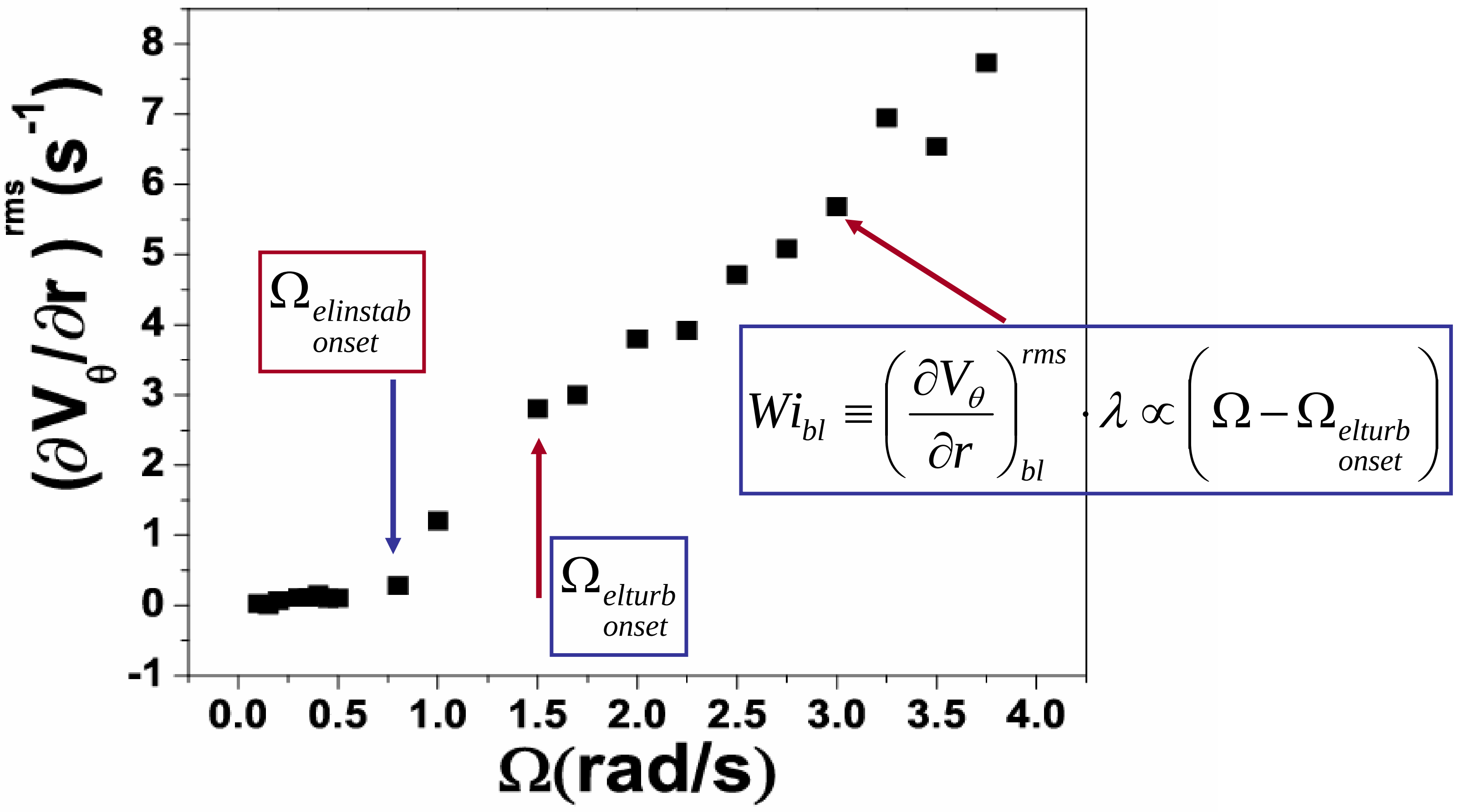


Probability distribution function of the normalized velocity gradients (from LDV)

$$Y(t) \equiv \frac{dV(t)}{dt} V_{av}^{-1};$$

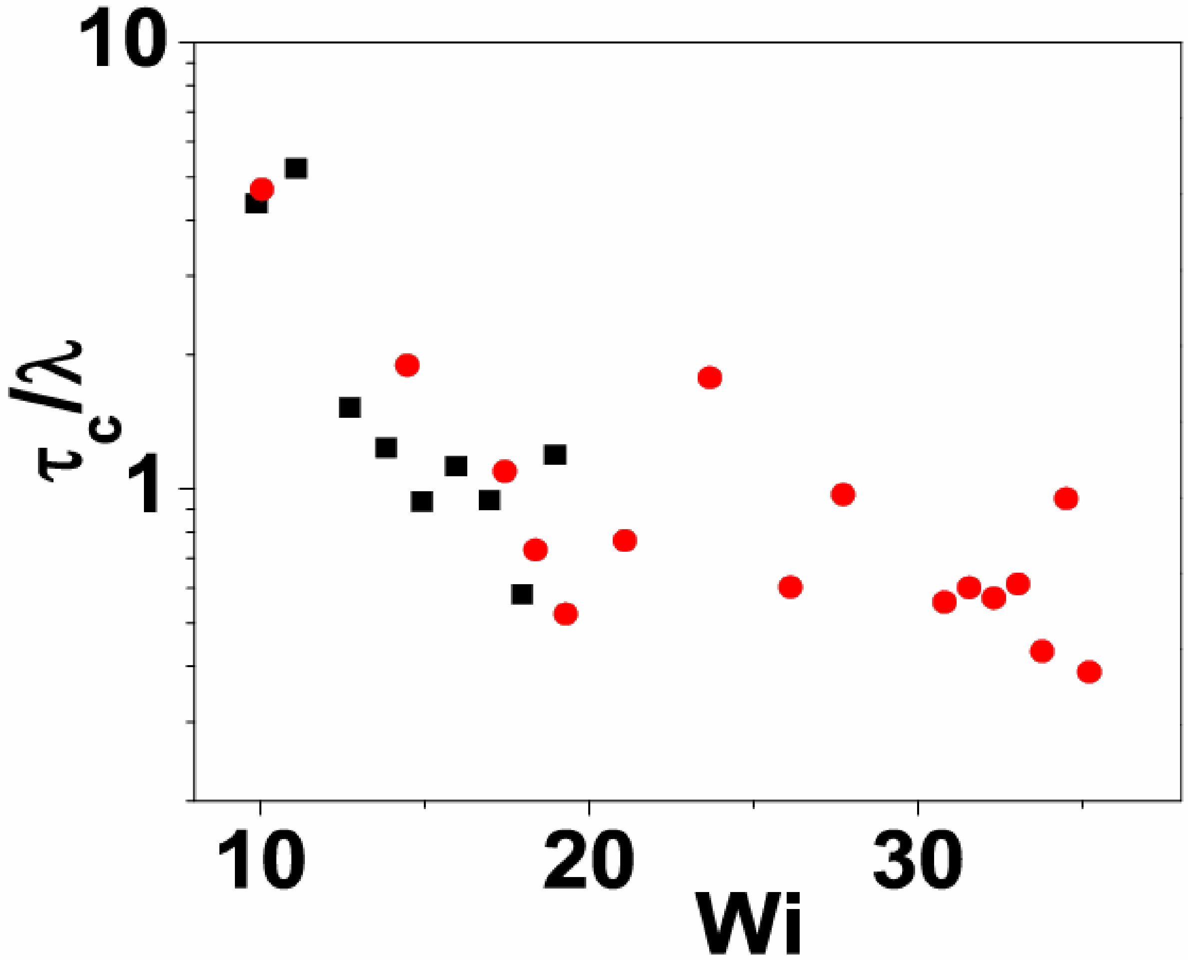


$Wi_{bulk} \equiv \omega_{rms} \cdot \lambda$ -normalized rms vorticity at different radial locations in the bulk of the cell from r/R=0.2 till 0.66

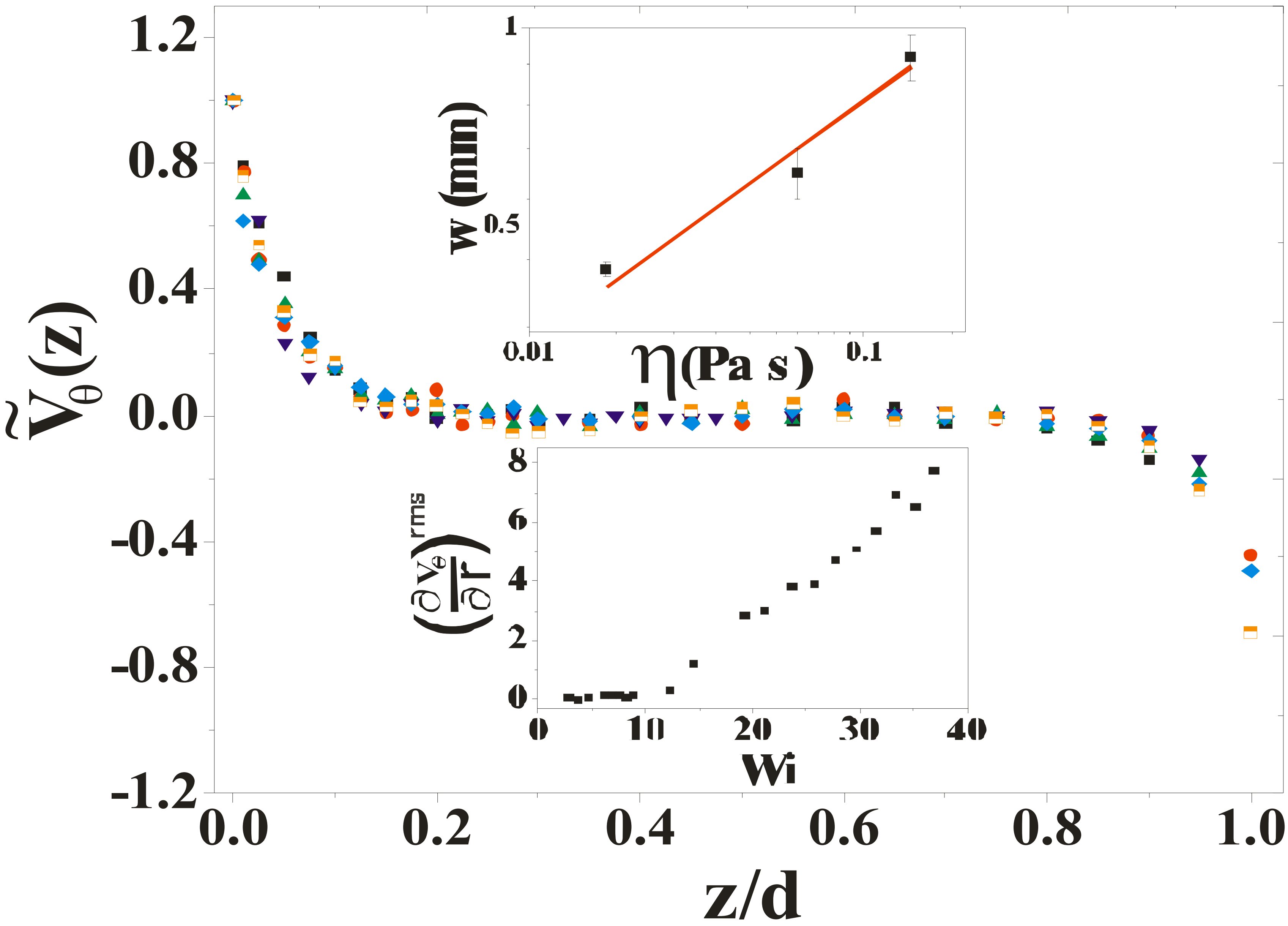


Dependence of rms of the velocity gradient in a boundary layer (slope of the de as a function of the rotation velocity.

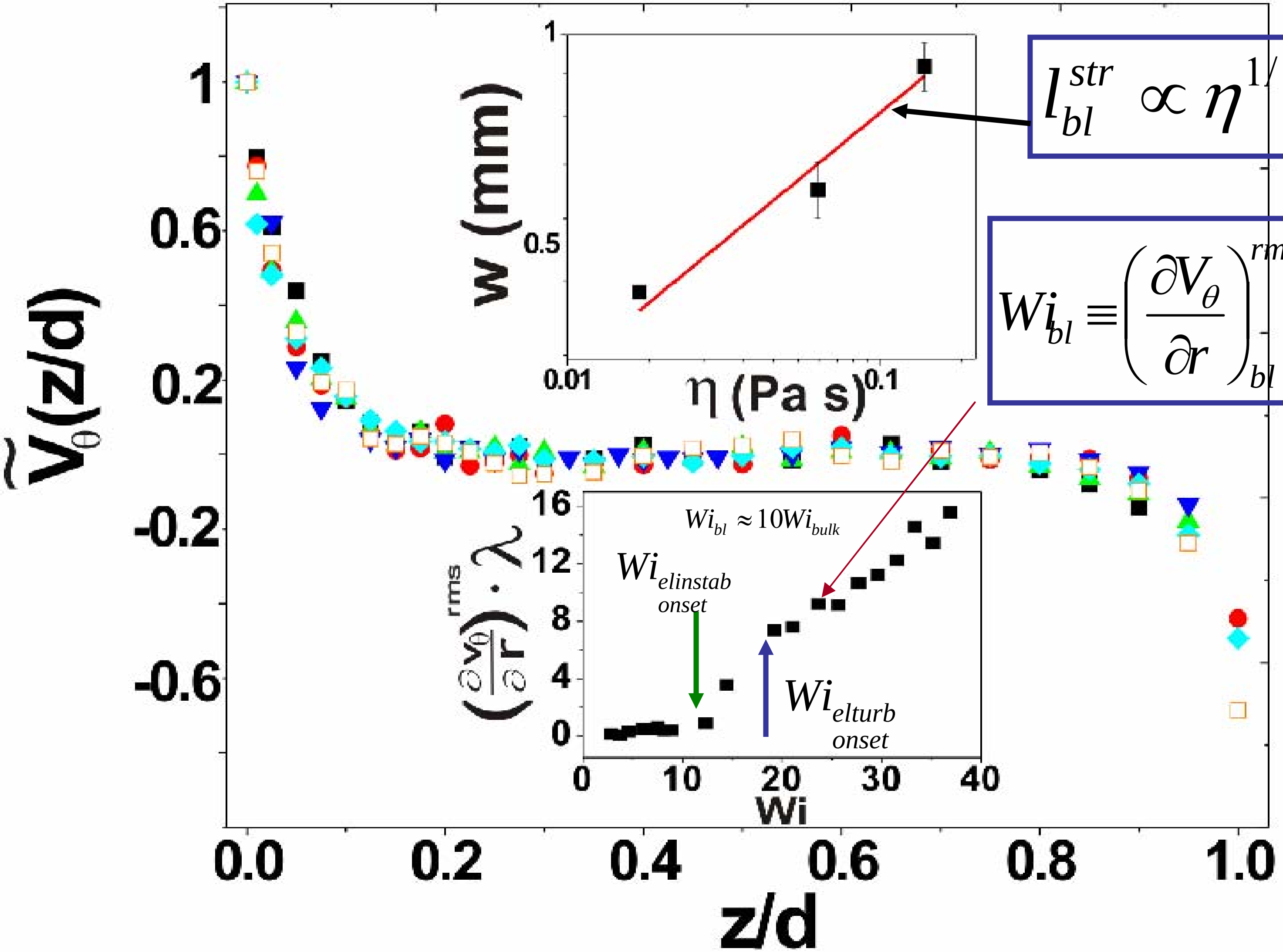
$$\boxed{Wi_{bl} \approx 10 Wi_{bulk}}$$



τ_c / λ -normalized velocity correlation time (two setups)



Normalized average azimuthal velocity profiles at different Wi; boundary layer width as a function of viscosity; rms of the velocity gradient in boundary layer.



$$l_{bl}^{str} \propto \eta^{1/4}$$

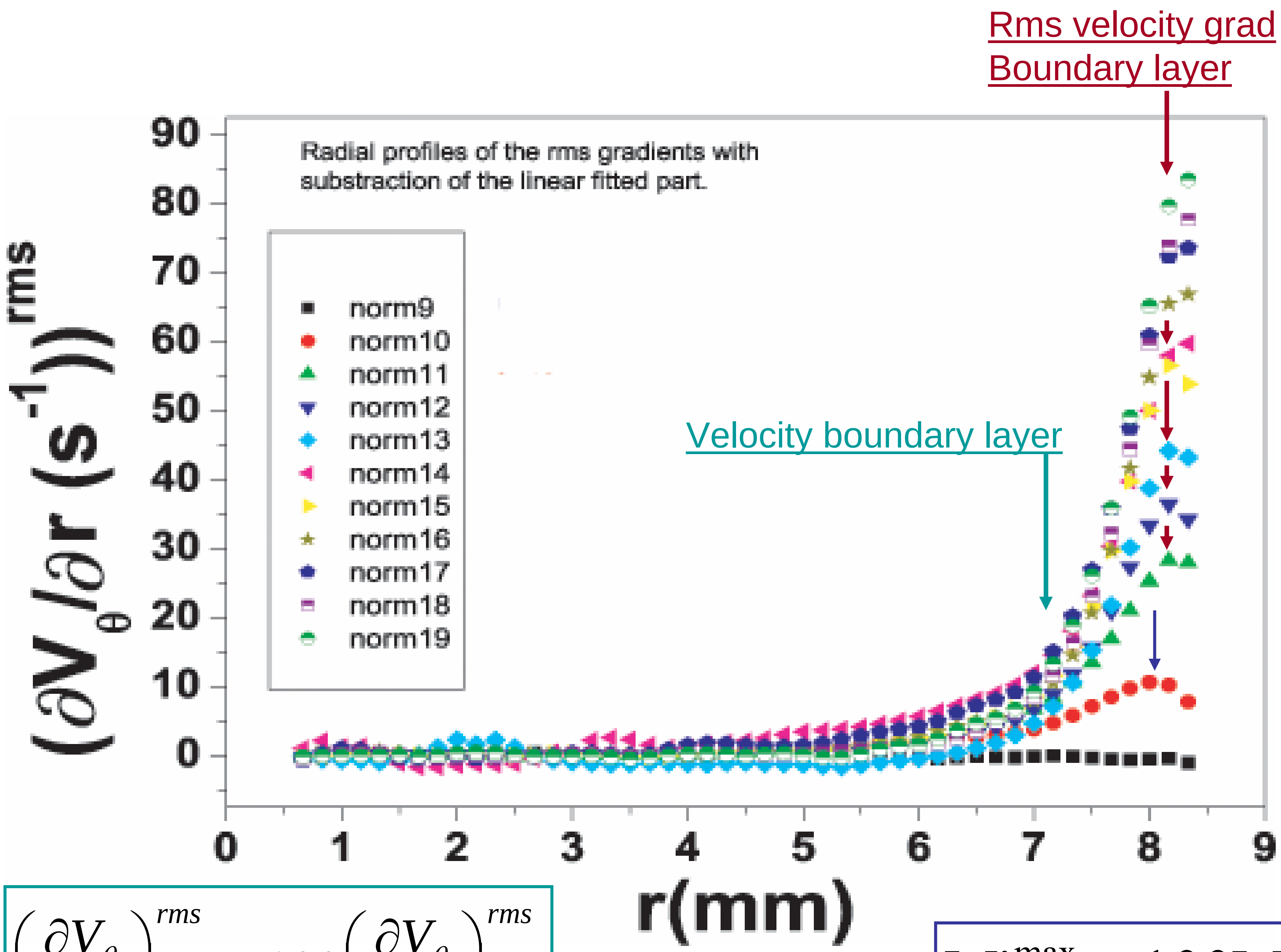
$$Wi_{bl} \equiv \left(\frac{\partial V_\theta}{\partial r}\right)_{bl}^{rms} \lambda \propto \left(Wi - Wi_{\substack{elturb\\onset}}\right)$$

analogy

$$l_{bl}^{ps} \propto D^{1/4}$$

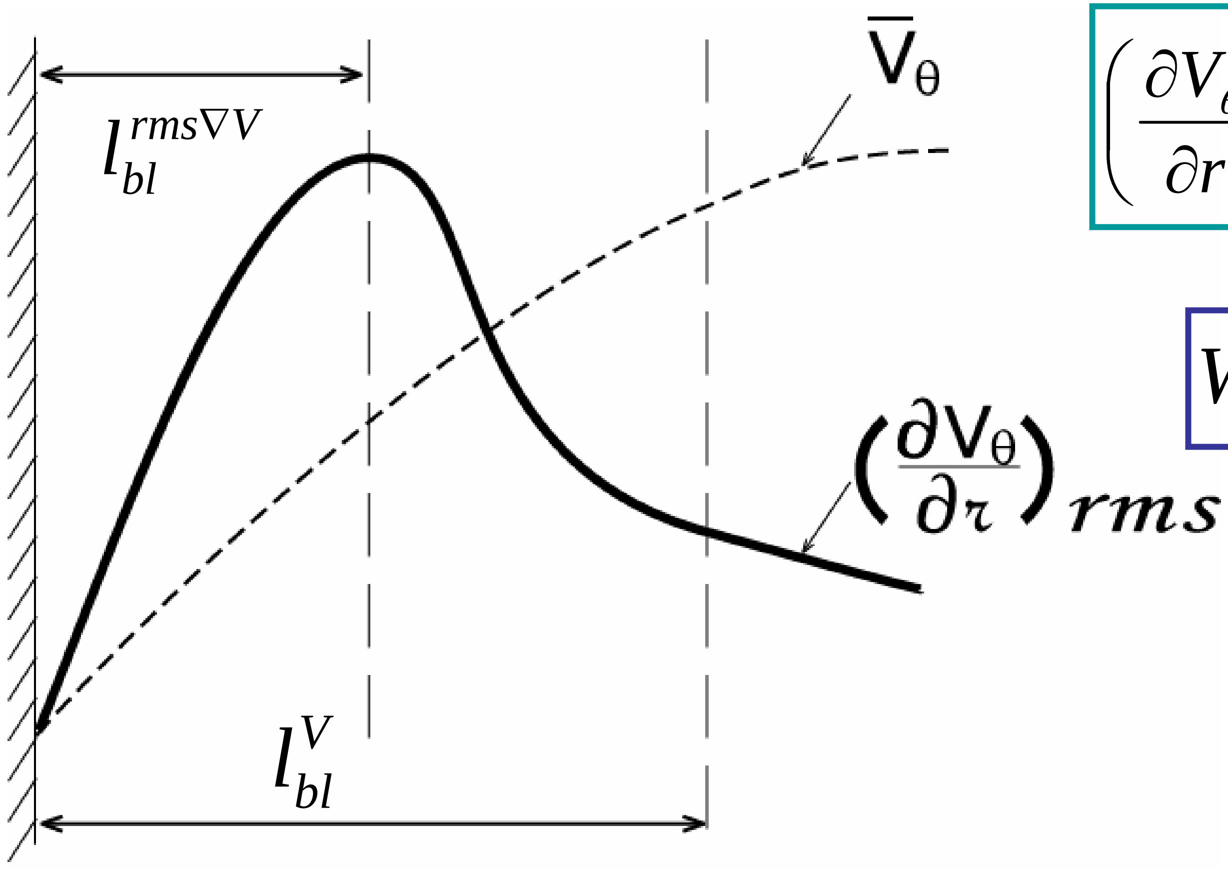
$$l_{bl}^{str} \propto \eta^{1/4}$$

Normalized average azimuthal velocity profiles at different Wi; boundary layer width as a function of viscosity; rms of the velocity gradient in boundary layer.



$$\left(\frac{\partial V_\theta}{\partial r}\right)^{rms}_{\max} \approx 100\left(\frac{\partial V_\theta}{\partial r}\right)^{rms}_{bulk}$$

$$Wi_{bl}^{\max} \approx 100\,Wi_{bulk}$$



$$\left(\frac{\partial V_\theta}{\partial r}\right)^{rms}_{\max} \approx 100\left(\frac{\partial V_\theta}{\partial r}\right)^{rms}_{bulk}$$

$$Wi_{bl}^{\max} \approx 100 Wi_{bulk}$$

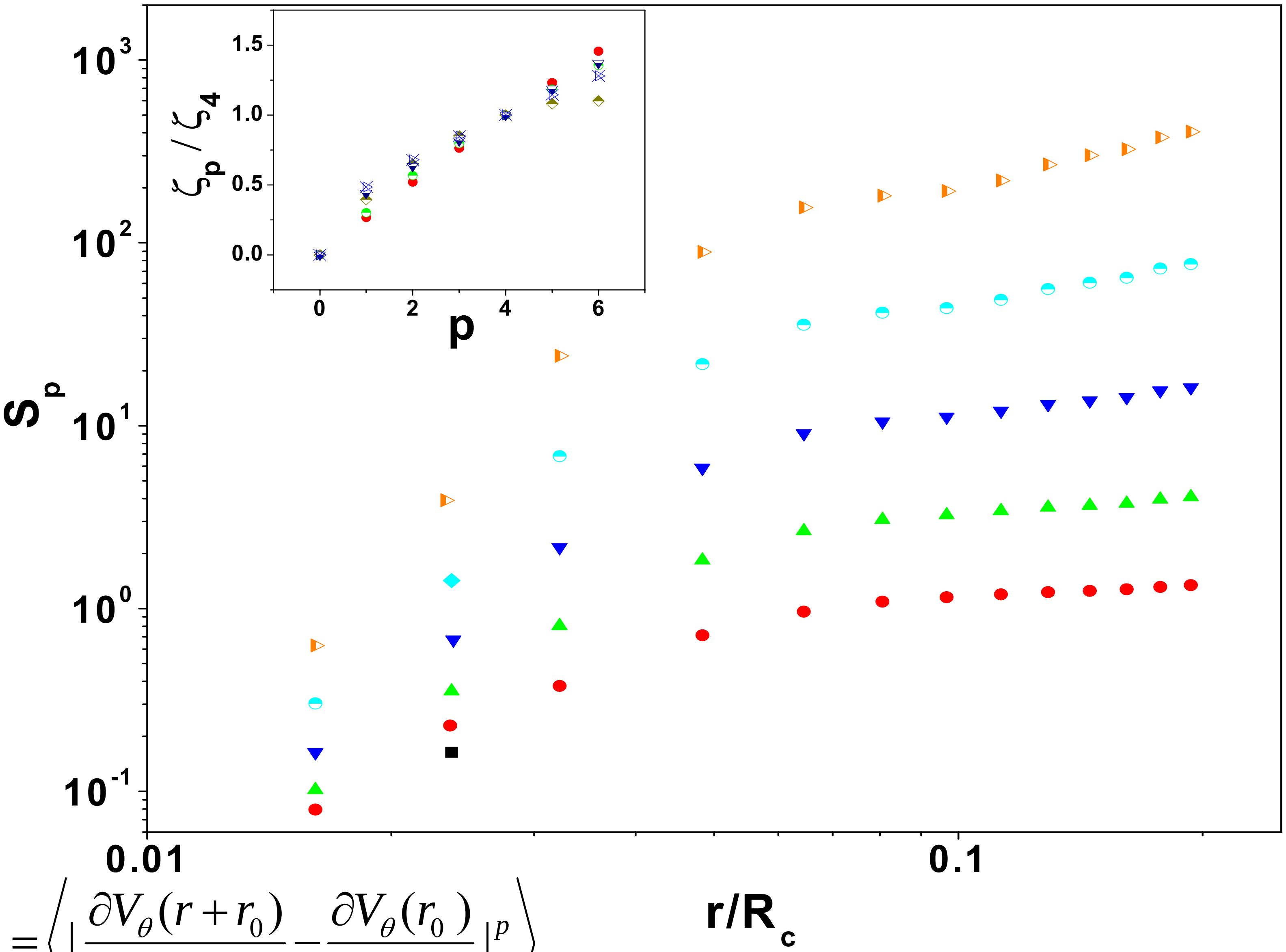
New characteristic spatial scale-boundary layer width

We consider elastic turbulent flow that occurs at scales below the dissipation scale in hydrodynamic turbulence.

So the scale can be related only to scales of elastic stresses:
In passive scalar problem $l_{bl}^{ps} \propto Pe^{-1/4}; Pe >> 1$; while in elastic turbulence

$$l_{bl}^{str} \propto \left(\frac{Wi_{bulk}}{Wi_{bl}^{\max}}\right)^{\alpha} << 1$$

$$l_{bl}^{ps} \propto D^{1/4}$$



$$S_p(r) = \left\langle \left| \frac{\partial V_\theta(r+r_0)}{\partial r} - \frac{\partial V_\theta(r_0)}{\partial r} \right|^p \right\rangle_{r_0}$$

Structure functions of vorticity and of injected power and its scaling exponents vs its order together with the same for the passive scalar structure functions

Role of elastic stress in statistical and scaling properties of elastic turbulence

Two main observations, namely saturation of $\left(\frac{\partial V_\theta}{\partial r}\right)_{bulk}^{rms}$ (that means elastic stress) and linear growth of $\left(\frac{\partial V_\theta}{\partial r}\right)_{bl}^{rms}$ (or elastic stress) in boundary layer can explain most of the results found experimentally.

First, this non-uniform distribution of elastic stress should result in non-uniform distribution of polymer stretching-the polymer molecules should be stretched considerably more in the boundary layer-we verify now this prediction experimentally.

Second, new length scale-boundary layer width –appears as the result of this non-uniform stress distribution: $l_{bl}^{str} \propto \left(\frac{Wi_{bulk}}{Wi_{bl}^{\max}}\right)^{\alpha} << 1$

Third, elastic stress from BL is intermittently injected into bulk that leads to skewness and exponential tail in PDF of injected power (torque) and exponential tails in PDF of velocity gradients in a bulk. Saturation of elastic stress in a bulk is consistent with scaling of $\frac{\overline{P}}{P_{lam}}$ observed.

This picture is in full analogy with the passive scalar problem in the Batchelor regime of mixing in a bounded container, where excess of tracer from BL is intermittently injected into bulk

Exploring further this analogy one can suggest that the boundary width is $l_{bl}^{str} \propto \eta^{1/4}$ that was observed experimentally. So

$$\boxed{l_{bl}^{str} \propto \left(\frac{Wi_{bulk}}{Wi_{bl}^{\max}} \right)^{\alpha} \eta^{1/4}}$$

The analogy with passive scalar is deepened by similarity in structure functions scaling. and direct relation to magneto-dynamo is also established.

Theory of Polymers Dynamics in Turbulent Flow: long-standing problem

(Lumley (1971),

E. Balkovsky, A. Fouxon, V. Lebedev, ***PRL* 84,** 4765 (2000)

M. Chertkov, ***PRL* 84**, 4761 (2000)

A. Celani, S. Musacchio, D. Vincenzi, ***J. Stat. Mech***. **118**, 529 (2005)

M. Chertkov, I. Kolokolov, V. Lebedev, K. Turitsyn, ***J. Fluid Mech.* 531**, 251 (2005)

K. Turitsyn, submitted to ***PRE*** (2005)

M. Martins, D. Vincenze, submitted to ***J. Fluid Mech***. (2005)

Polymer molecule in turbulent or chaotic flow

R<η-dissipation length.

So a polymer molecule is immersed into spatially smooth and temporally random velocity field (similar to Batchelor problem of passive scalar mixing).

Thus, dynamics of stretching is governed by the velocity gradient only

$$\vec{V}(r,t) = \frac{\partial V_i}{\partial r_j}(t) \cdot \vec{r}$$

$\frac{\partial V_i}{\partial r_j}$ -rate of deformation in Lagrangian coordinates randomly varies in time.

Two nearby fluid elements diverge exponentially on average:

$$\langle R(t) \rangle = R(0) \exp(\gamma t)$$

Since polymer molecule follows deformation of a fluid element, it can be stretched significantly even in a random flow (γ is the Lyapunov exponent and defines the rate of stretching of a fluid element)

Elastic turbulence and statistics of polymer molecules extension.

$R_0 << R << R_{max}$

Dynamic equation for the end-to-end vector **R**=R**n:**

$$\frac{d}{dt}R_i = R_j \nabla_j V_i - \frac{R_i}{\tau}$$

Probability distribution function of the molecule size R:

$$\mathrm{P}(R_i) \propto R_0{}^{\alpha} R_i{}^{-\alpha-1} \qquad \text{where} \qquad \alpha \propto (\tau^{-1} - \gamma)$$

At $\alpha<0$ the majority of molecules is strongly stretched.

At $\alpha>0$ the majority of molecules has nearly equilibrium size.

α changes sign at $\gamma\tau=1$. At $\gamma\tau<1 \Rightarrow \alpha>0$; at $\gamma\tau>1 \Rightarrow \alpha<0$.

*Thus, Wi'=$\gamma\tau$ plays a role of a local Weissenberg number for a random flow and the condition α=0 can be interpreted as the criterion for the coil-stretch transition in a random flow that occurs at **Wi'=1**.*

Numerical Simulations

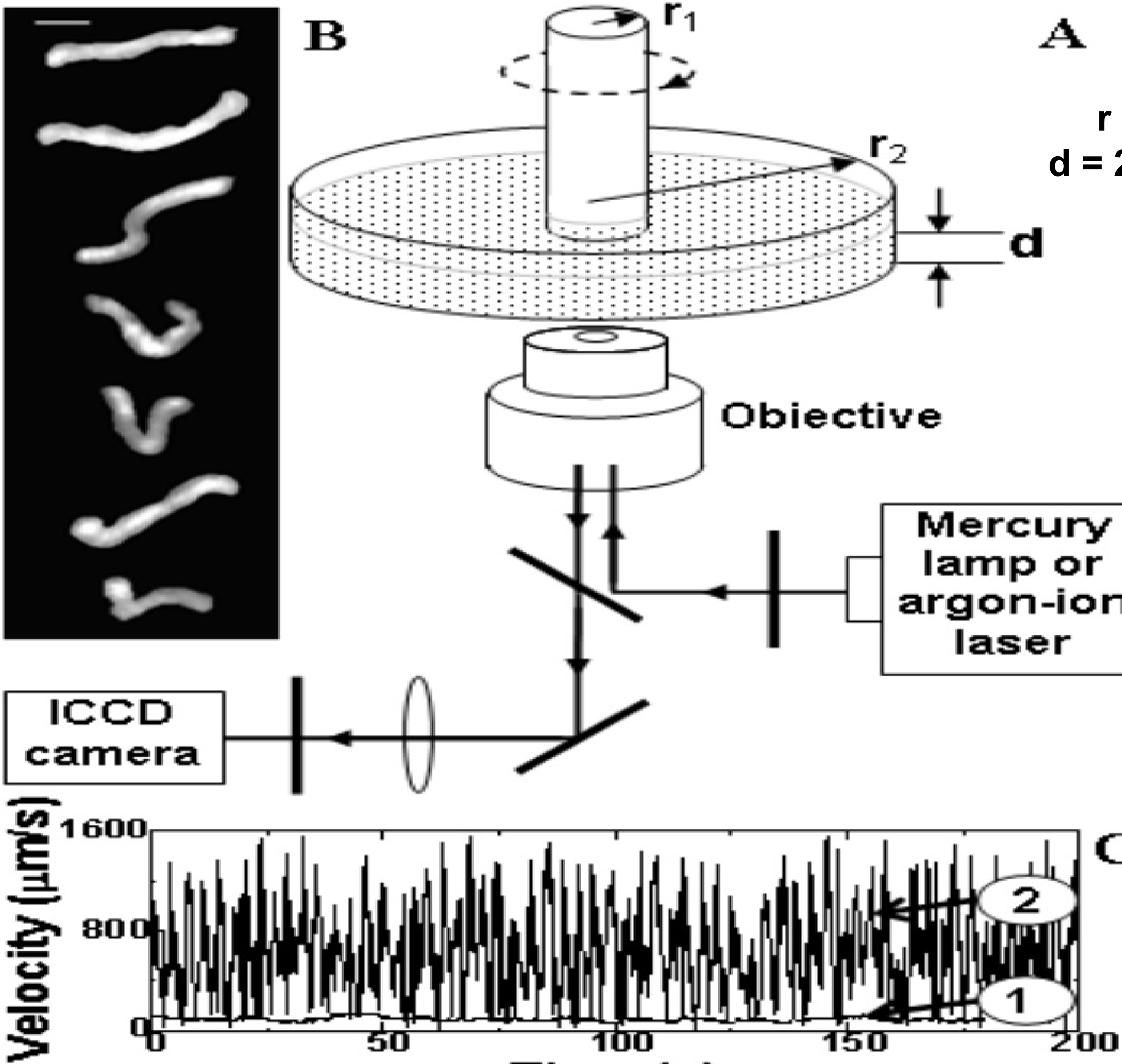
B. Eckhardt, J.Kronjager, J.Schumacher, ***Comput. Phys. Commun.*** **147**, 538 (2002)

G. Boffetta, A. Celani, S. Musacchio, ***PRL*** **91**, 034501 (2003)

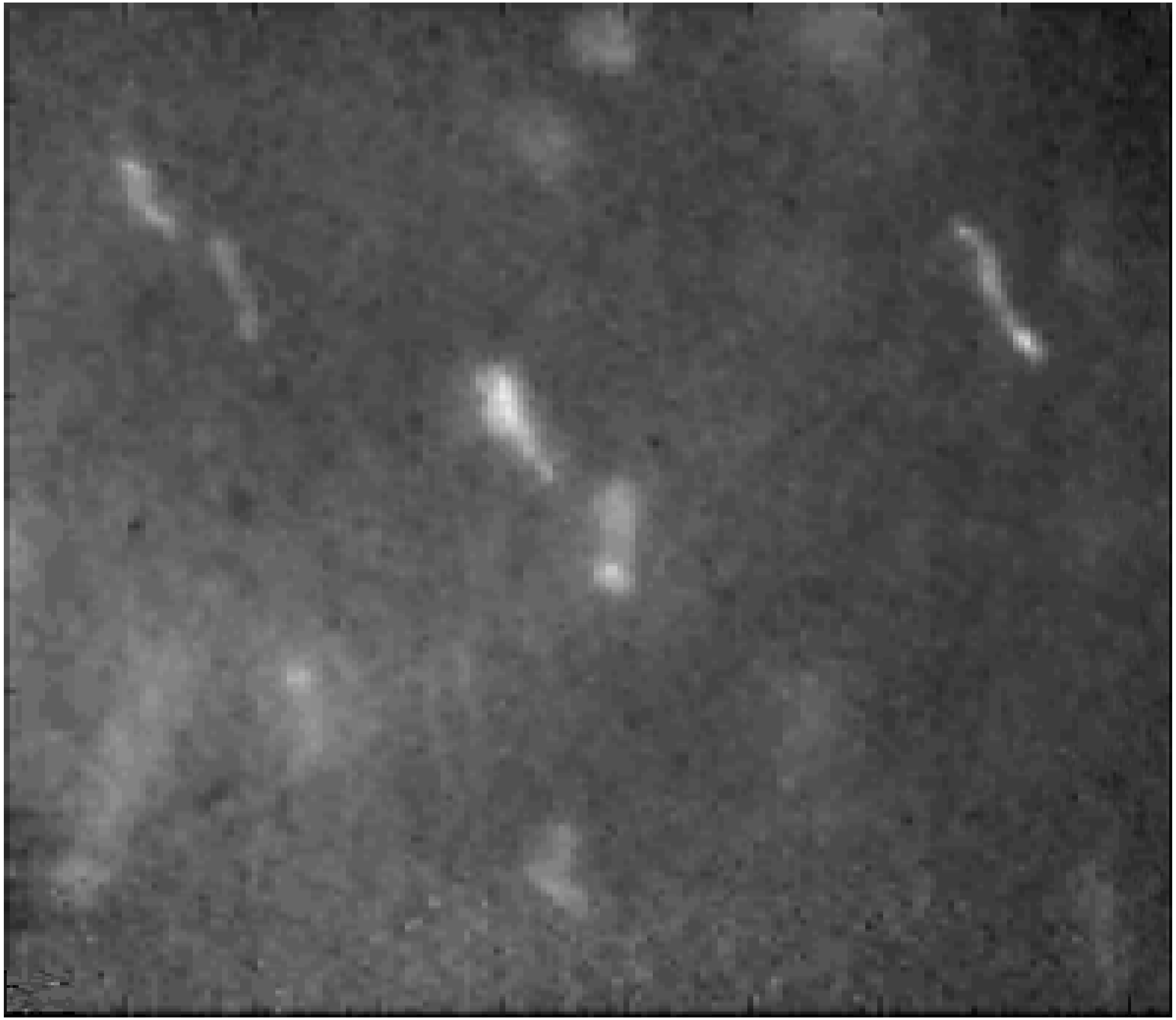
A. Celani and A. Puliafito, private communication and to be published (2005)

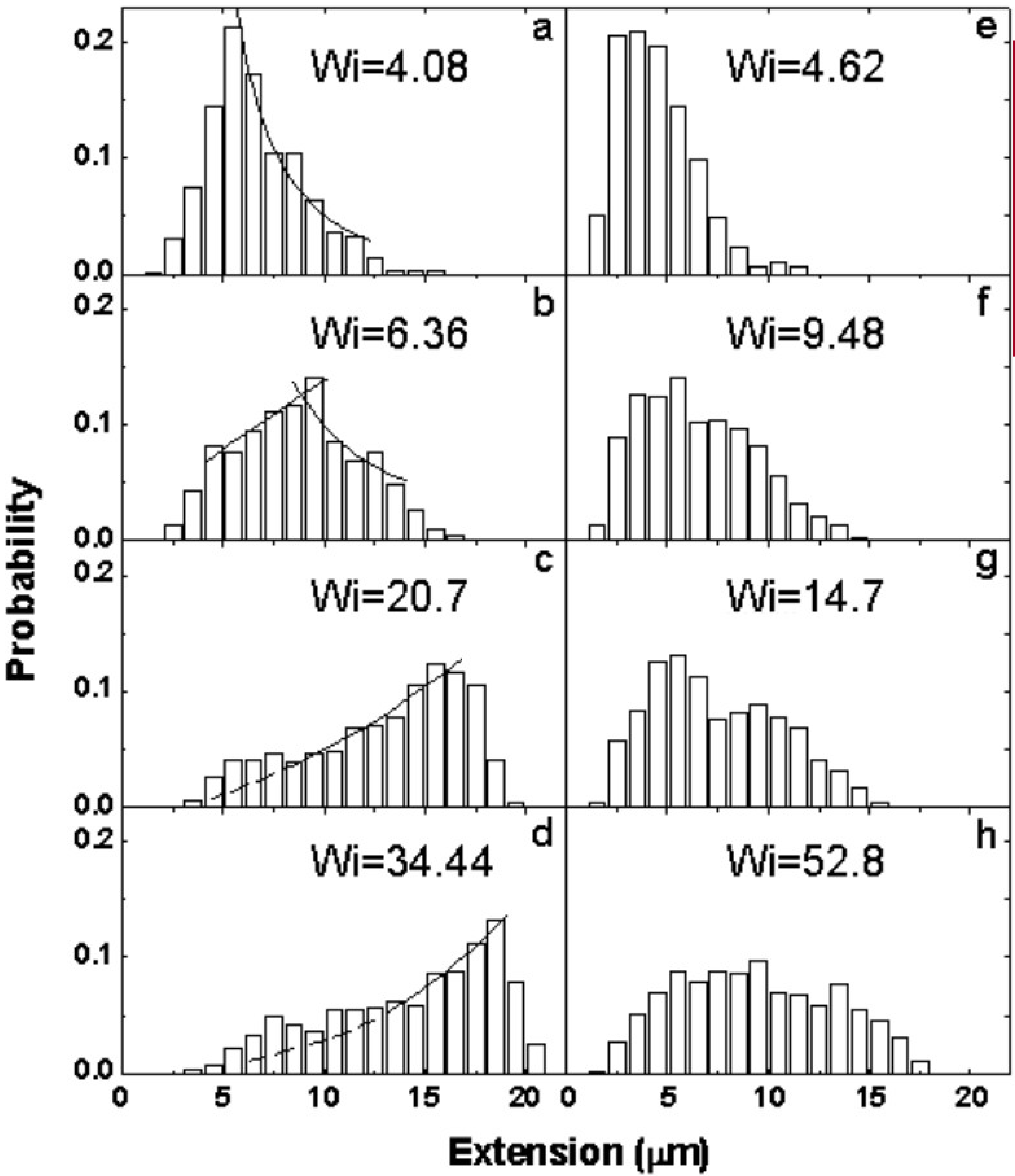
Single Polymer Dynamics: coil-stretch transition in a random flow.

S. Geraschenko, C. Chevallard, V. Steinberg,
Europhys. Lett. **71**, 221 (2005)

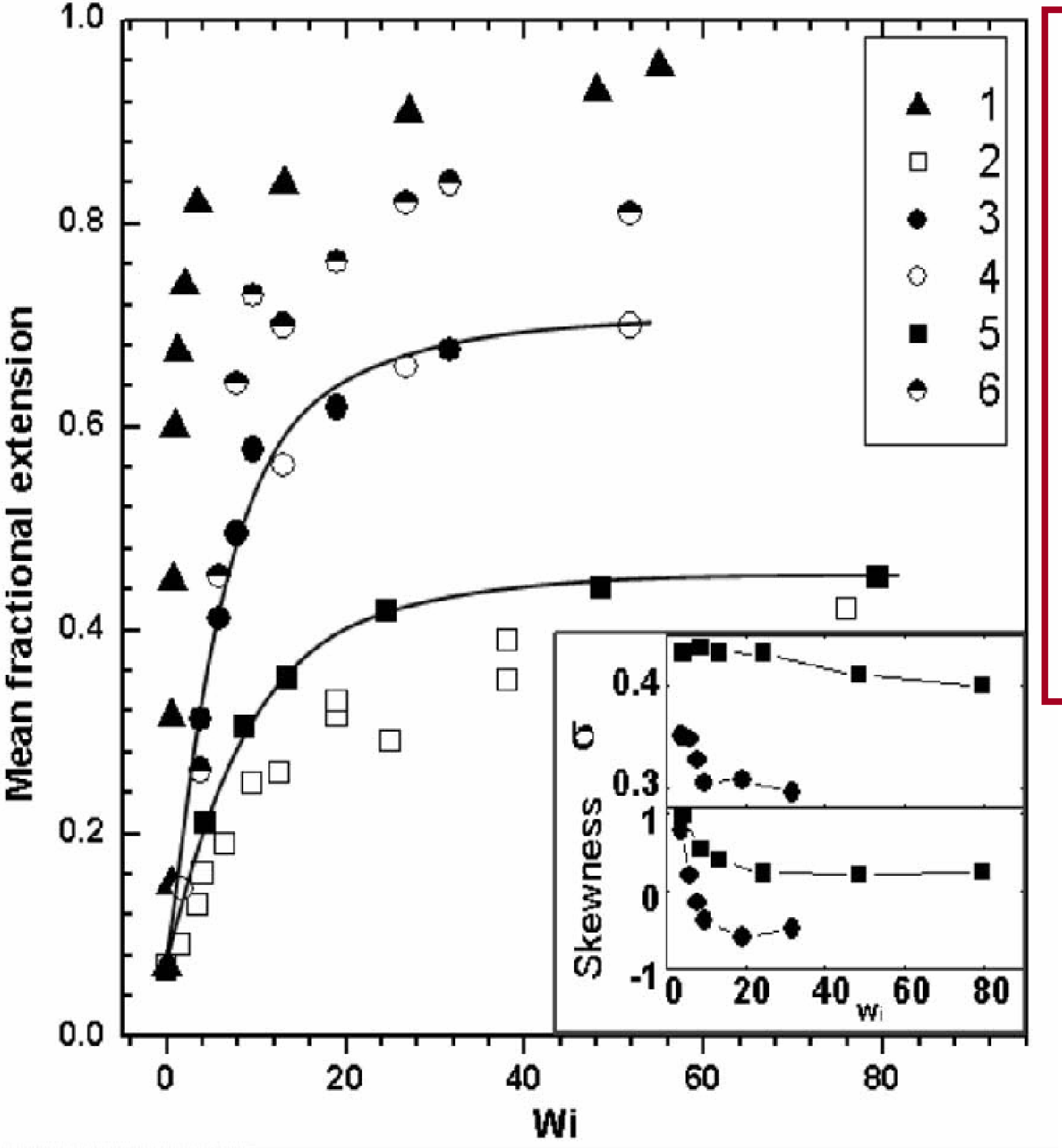


B
A
r_1
r_2
r = 1mm
d = 200 μm
d
Objective
Mercury lamp or argon-ion laser
ICCD camera
C
Velocity (μm/s)
1600
800
0
0
50
100
150
200
Time (s)
2
1

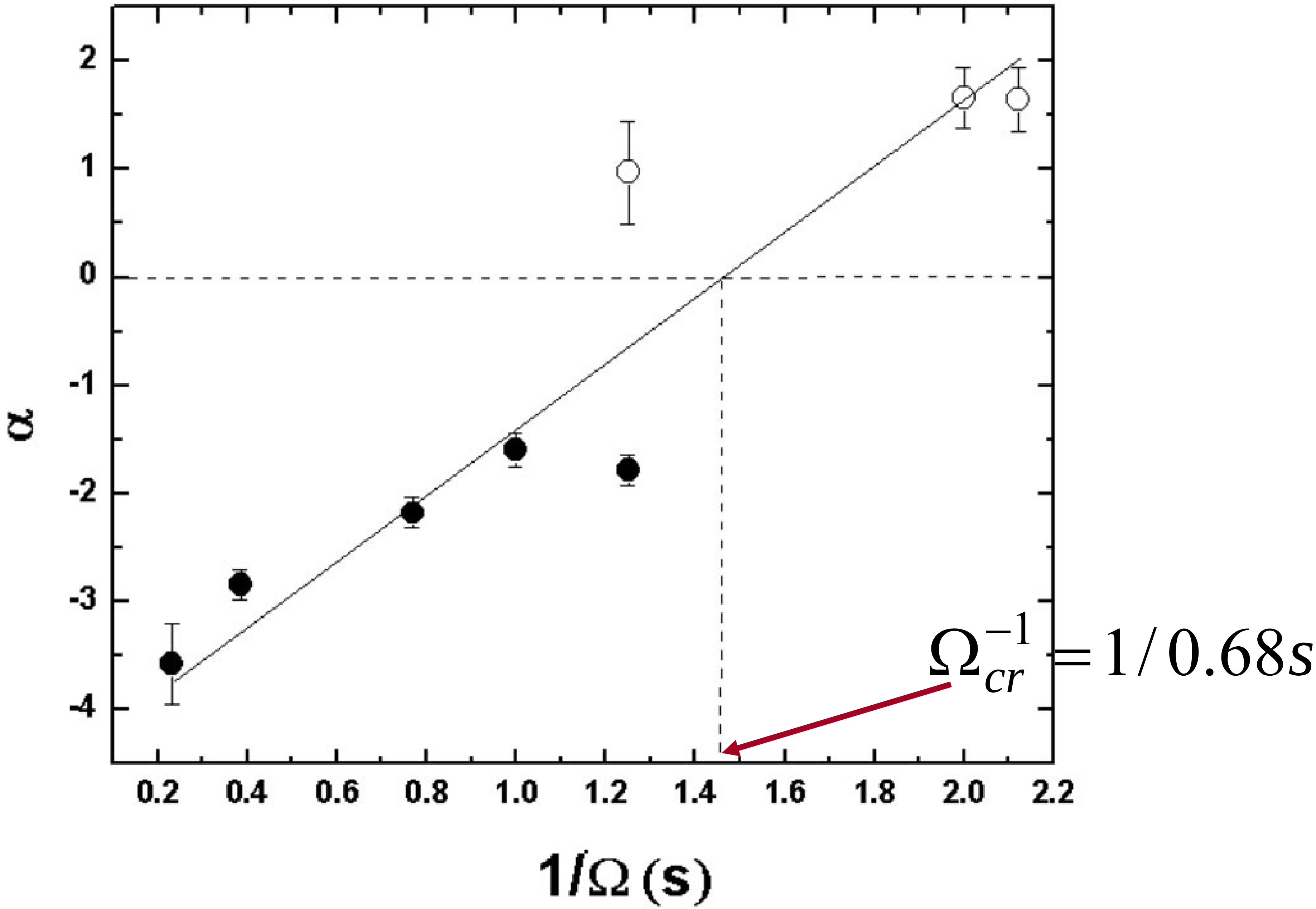




PDF of polymer extension at various Wi in random (left) and shear (right) flows
Solid lines are fits by power law $P(R_i) \propto R_i^{-\alpha-1}$ ***with*** α ***defined from the fits.***



(1)-elongational flow
(S.Chu et al, (1997));
(2)-plane shear flow
(S.Chu et al, (1999));
(3,4)-random flow in λ-DNA
and PAAm solutions;
(5)-shear flow in both solutions
(6)-fractional extensions in random flow corresponding to PDF's maximum



Black dots are the data above the coil-stretch transition, open circles- below

Finite-time Lyapunov exponents

- Probability of finite-time Lyapunov exponents (FTLE) is defined as

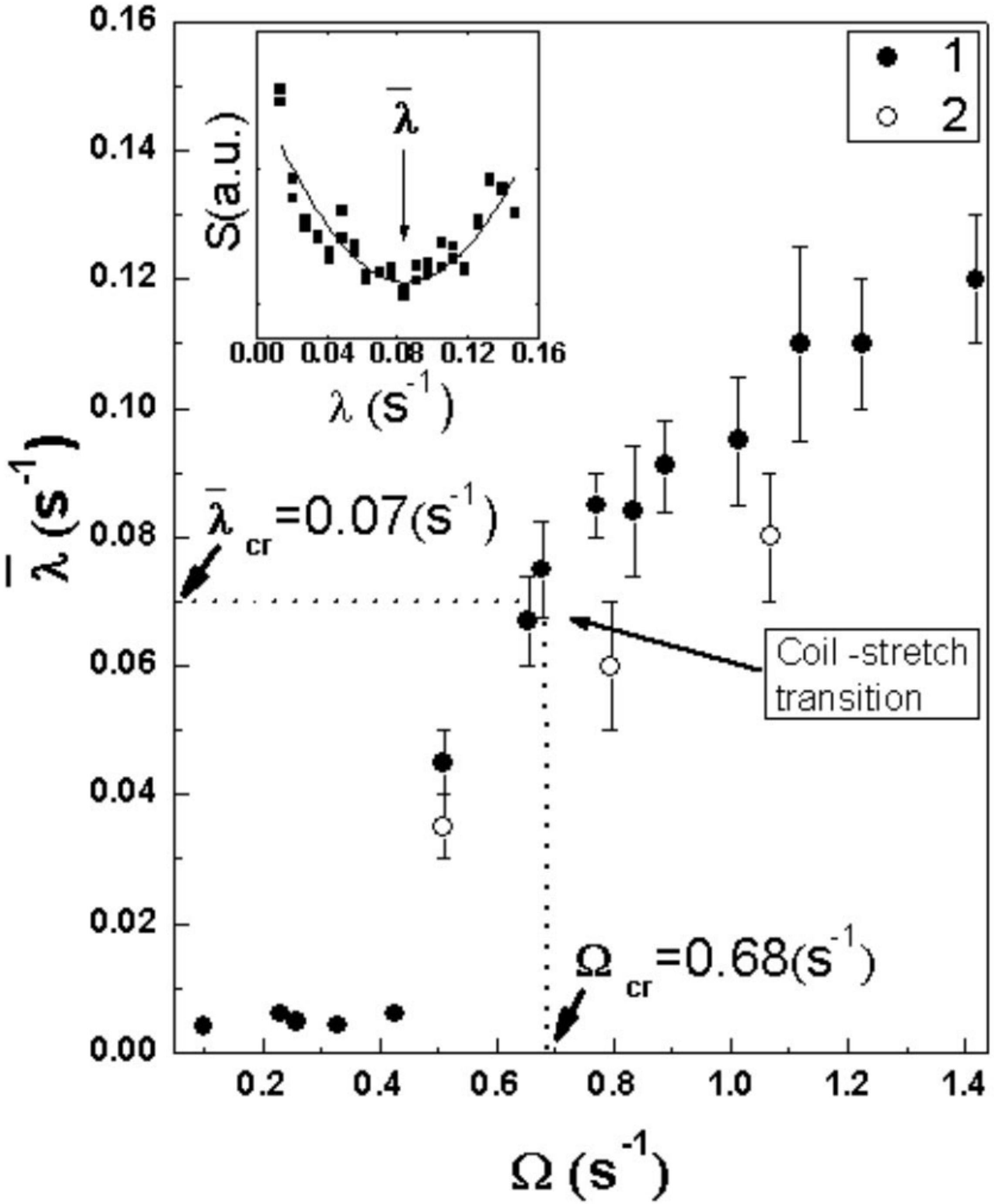
$$P(\gamma, t_i) \propto \exp[-t_i S(\gamma)]$$

- By rescaling one can collapse the probability distribution functions for different $t_i < t_{corr}$ on one curve

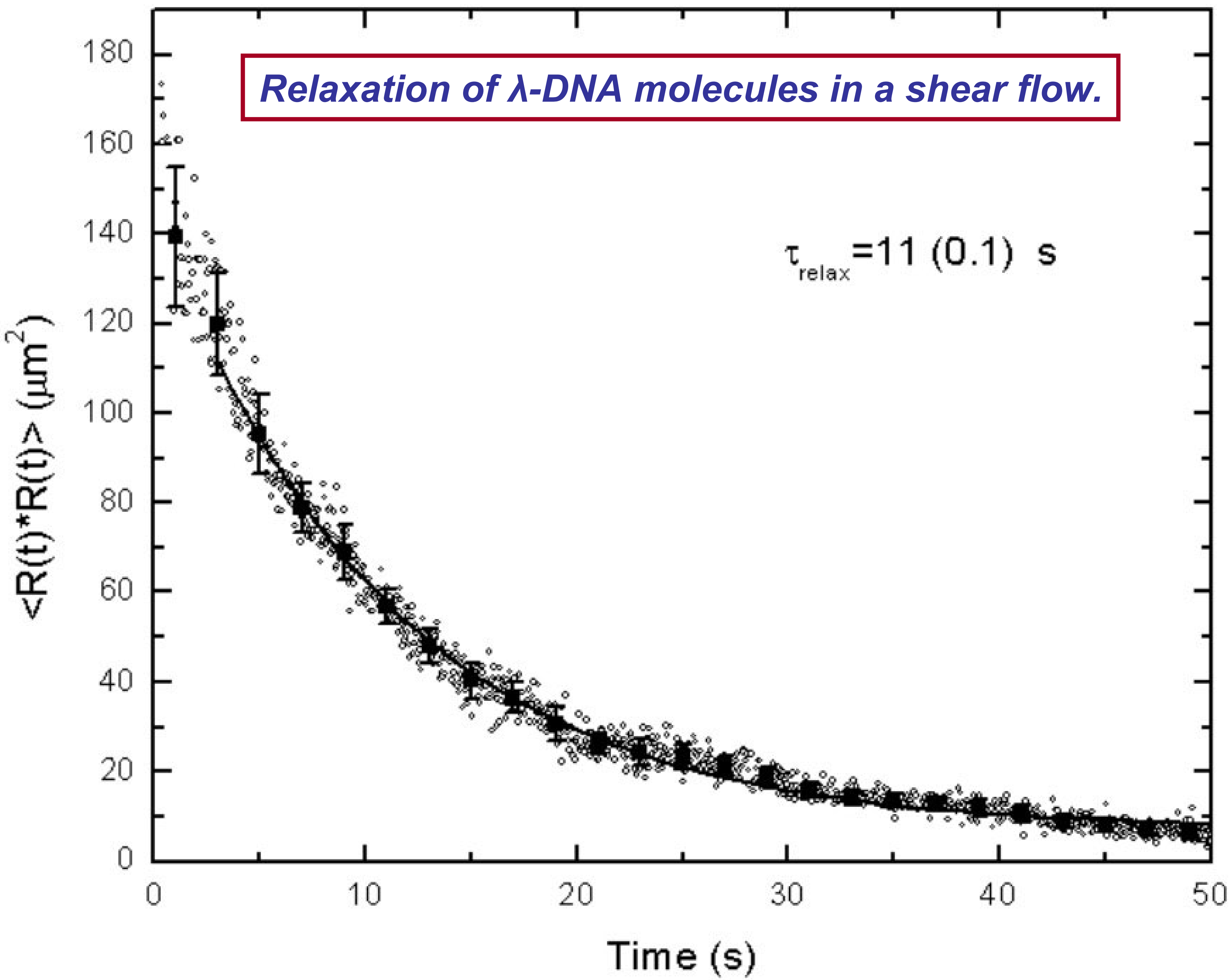
$$S(\gamma) \propto -\frac{\ln P(\gamma, t_i)}{t_i}$$

where S(λ) is the Cramer rate function.

- Minimum of $S(\gamma)$ determines the average Lyapunov exponent $\overline{\lambda}$ for a given rotation rate (i.e. $\overline{\lambda}(t_i) \cong const$ for all PDFs scaled).



Average Lyapunov exponent $\bar{\lambda}$ as a function of Ω for (1) PAAm and (2) λ-DNA solutions.
Inset: Cramer rate function at Ω=0.83 1/s.


Relaxation of λ-DNA molecules in a shear flow.
τrelax=11 (0.1) s
<R(t)*R(t)> (μm2)
Time (s)
0
20
40
60
80
100
120
140
160
180
0
10
20
30
40
50

Criterion of the coil-stretch transition:

$$\bar{\lambda}_{cr} = 0.07 \pm 0.015 \quad 1/\mathrm{sec}$$

$$\tau = 11 \pm 0.1 \ \mathrm{sec}$$

The experimental value for the critical Weissenberg number

$$\widetilde{Wi} \equiv \bar{\lambda}_{cr}\tau = 0.77 \pm 0.2$$

and should be compared with theoretical prediction:

$$\widetilde{\mathrm{Wi}} = \mathbf{1}$$

Conclusions:

1. The role of elastic stresses in the statistical and scaling properties of elastic turbulence is investigated experimentally.

2. The analogy between the elastic turbulence and the passive scalar turbulent advection is fully supported by our data on global injected power, velocity gradients and RMS of their fluctuations. Here, the elastic stresses play the role of the passive scalar.

3. The global injected power scales as $\overline{P}/P_{lam} \propto Wi^{0.49\pm0.05}$ which is consistent with the saturation of ω^{rms} and elastic stresses in a bulk.

4. $\left(\partial V_\theta / \partial r\right)^{rms}$ (and thus the stretching of polymer molecules) increases linearly with Wi in the boundary layer and saturates in the bulk, suggesting that the elastic energy is randomly and intermittently injected at large scales.

5. Exploring further the passive scalar analogy, we suggest the following scaling for the boundary layer width: $l_{bl}^{str} \propto (Wi_{bulk} / Wi_{bl}^{\max})^{\alpha} \eta^{1/4}$

Early related publications:

[1] A. Groisman and V. Steinberg, Nature **405**, 53 (2000).
[2] A. Groisman and V. Steinberg, Nature **410**, 905 (2001).
[3] A. Groisman and V. Steinberg, New J. Phys. **6**, 29 (2004).
[4] T. Burghelea, E. Segre, I. Bar-Joseph, A. Groisman, and V. Steinberg, Phys. Rev. E **69,** 066305 (2004).
[5] T. Burghelea, E. Segre, and V. Steinberg, Phys. Rev. Lett. **92**, 164501 (2004).

Recent related publications:

[1] T. Burghelea, E. Segre, and V. Steinberg, Europhys. Lett.**, 68,** 529 (2004).
[2] T. Burghelea, E. Segre, and V. Steinberg, Phys. Fluids, **17**, 103101 (2005).
[3] T. Burghelea, E. Segre and V. Steinberg, *“Role of elastic stresses in statistical and scaling properties of Elastic Turbulence”* , submitted to PRL (2005).
[4] V. Steinberg, “*Elastic Turbulence in Visco-elastic Flows*”, review, Ch. C2.3, in “Springer Handbook of Experimental Fluid Mechanics”, 2005.
[5] S. Gerashchenko, C. Chevallard, and V. Steinberg, Europhys. Lett.**, 71**, 221 (2005).

Theory of Elastic Turbulence

E. Balkovsky, A. Fouxon, V. Lebedev, ***PRE* 64**, 056301 (2001)
A. Fouxon, V. Lebedev, ***Phys. Fluids* 15**, 2060 (2003)

A crucial step towards a theoretical description of ET was to relate $\tau_p = \tau_{ik} = B_i B_k$ to the linearly decaying passive scalar problem:

$$\partial_t \vec{B} + \left(\vec{V} \cdot \nabla\right)\vec{B} = \left(\vec{B} \cdot \nabla\right)\vec{V} - \vec{B} / \lambda$$

The equation above, complemented by the equation of motion written for Re<<1:

$$\nabla P = \rho\left(\vec{B} \cdot \nabla\right)\vec{B} + \eta\nabla^2\vec{V}$$

and by the boundary conditions, leads to ELASTIC INSTABILITY at $Wi > 1$.

At Wi>Wic, the instability eventually results in chaotic, statistically steady dynamics. In the chaotic flow the velocity fluctuations dominate on the scale of the system size, and the dynamics are determined by non-linear interaction of modes with scales of the order of the system size.
Thus, the elastic stress on small scales can be estimated as:

$$\tau_p = B^2 \propto \nu\nabla_i V_j \propto \frac{\nu}{\lambda}$$