THE PRIMORDIAL PERTURBATION SPECTRUM

Genuine quantum-gravitational effect

Defined at the beginning of the FRWRD-stage ($\lambda > c/H$)

$\downarrow$

$\cdots \rightarrow dS \rightarrow$ FRWRD $\rightarrow$ FRWMD $\rightarrow \widetilde{dS} \cdots \rightarrow$

"primordial dark energy" — radiation — matter — present dark energy

1. Type of metric perturbations

Growing (or, quasi-isotropic) mode of scalar (adiabatic) perturbations

Consequences:

a) standing acoustic waves for $\lambda < c/H$ at the FRWRD stage $\rightarrow$ acoustic

(1970) peaks in CMB TT, EE and TE spectra,
(1965) Sakharov oscillations in $P(k)$

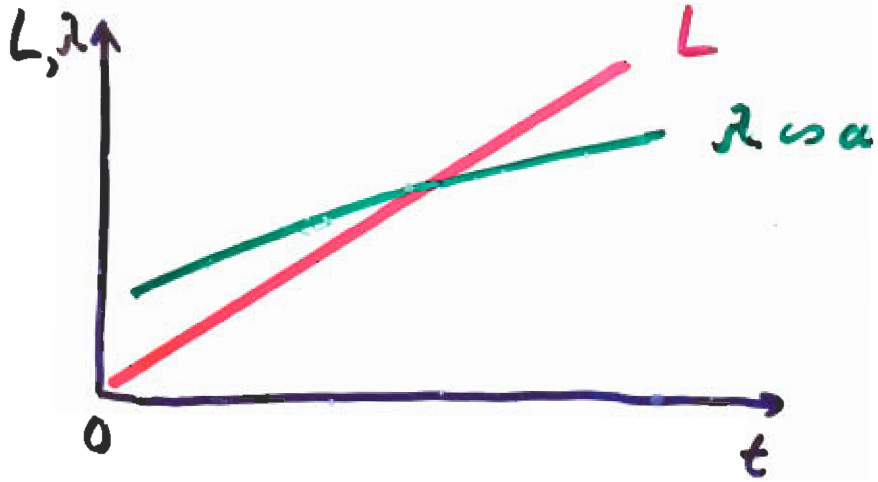
b) $\Psi_v \propto \Phi$, $\vec{a} \parallel \vec{v}$ at the FRWMD stage in the linear regime

All this has been observed!

Tests of adiabatic (curvature) perturbations – not restricted to inflationary model only, valid in any model where the Universe is isotropic and homogeneous up to the moment when $\lambda_{min} \sim c/H$ and perturbations are adiabatic ones

Problem of initial conditions

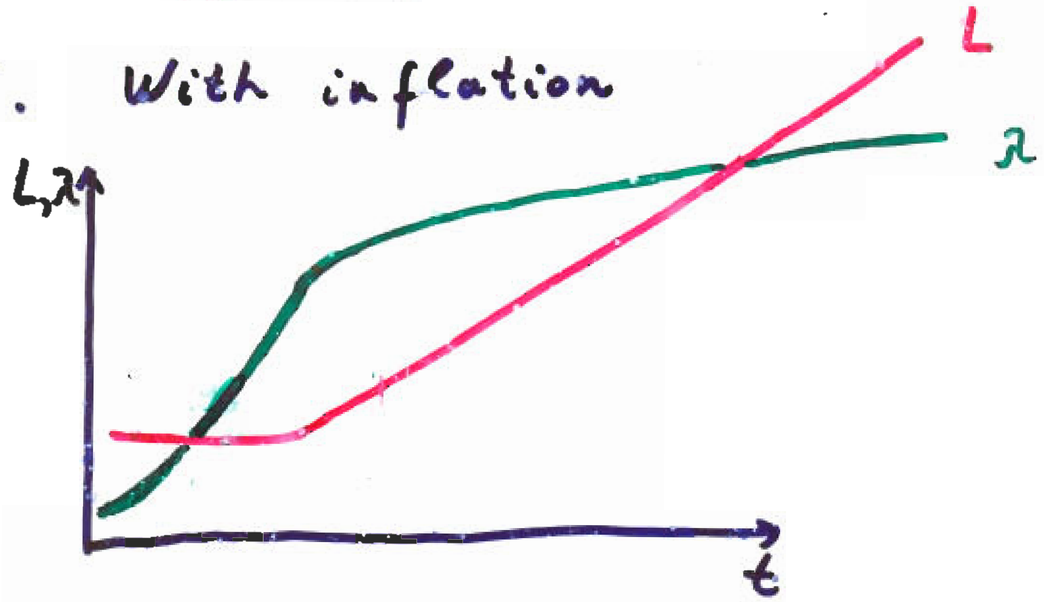
I. Without inflation



$L \sim \left(\frac{\dot{a}}{a}\right)^{-1}$

No unique principle to choose initial conditions

II. With inflation



Initial conditions may be chosen uniquely (due to the symmetry of the initial state)

After that, an inflationary model gives predictions for the present-day state of the Universe that may or may not agree with observational data

$$\boxed{\ln \frac{a_f}{a_{in}} > 70 - \frac{1}{2} \ln \frac{1}{H_f l_{pl}}}$$

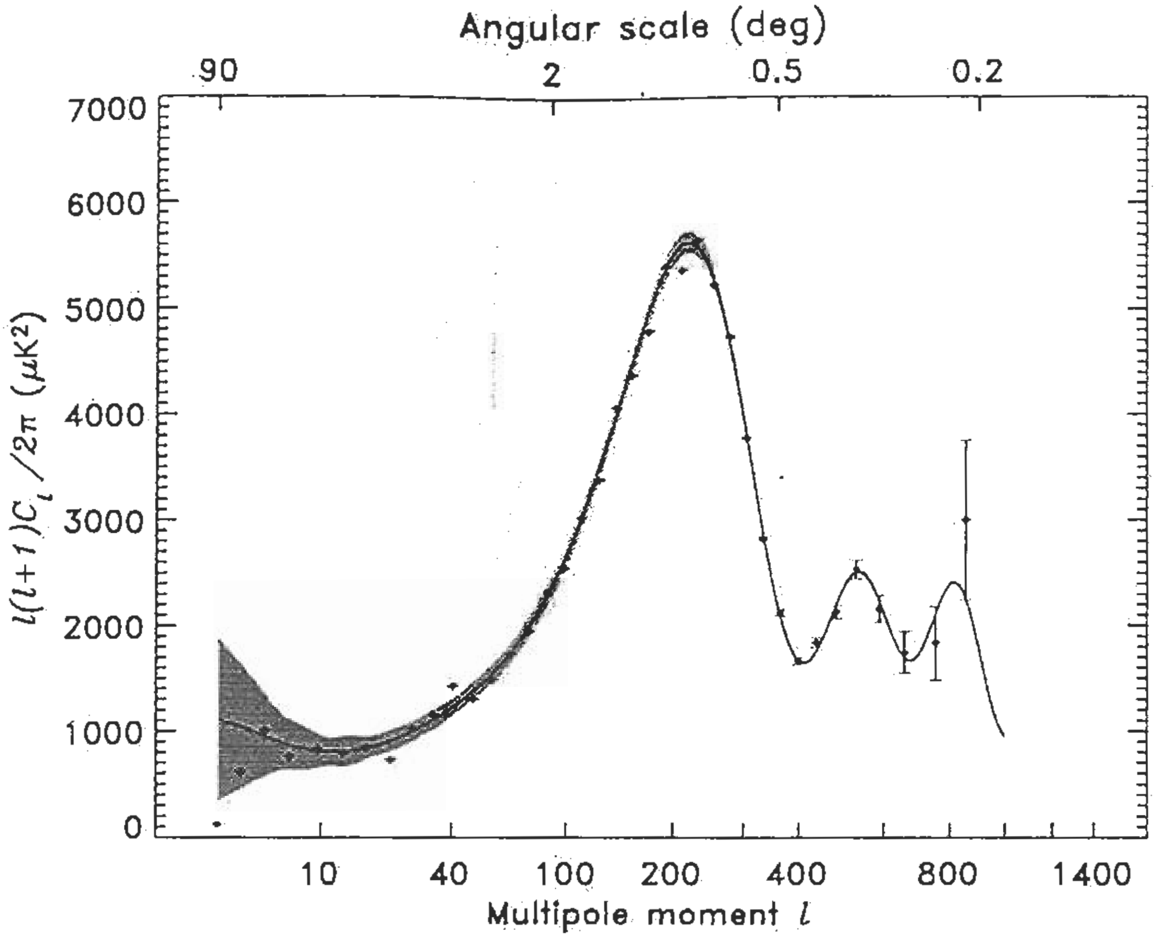


Fig. 8.— The final angular power spectrum, $l(l+1)C_l/2\pi$, obtained from the 28 cross-power spectra, as described in §5. The data are plotted with 1σ measurement errors only which reflect the combined uncertainty due to noise, beam, calibration, and source subtraction uncertainties. The solid line shows the best-fit ΛCDM model from Spergel et al. (2003). The grey band around the model is the 1σ uncertainty due to cosmic variance on the cut sky. For this plot, both the model and the error band have been binned with the same boundaries as the data, but they have been plotted as a splined curve to guide the eye. On the scale of this plot the unbinned model curve would be virtually indistinguishable from the binned curve except in the vicinity of the third peak.

M. Tegmark et al., ApJ 606, 702 (2004)

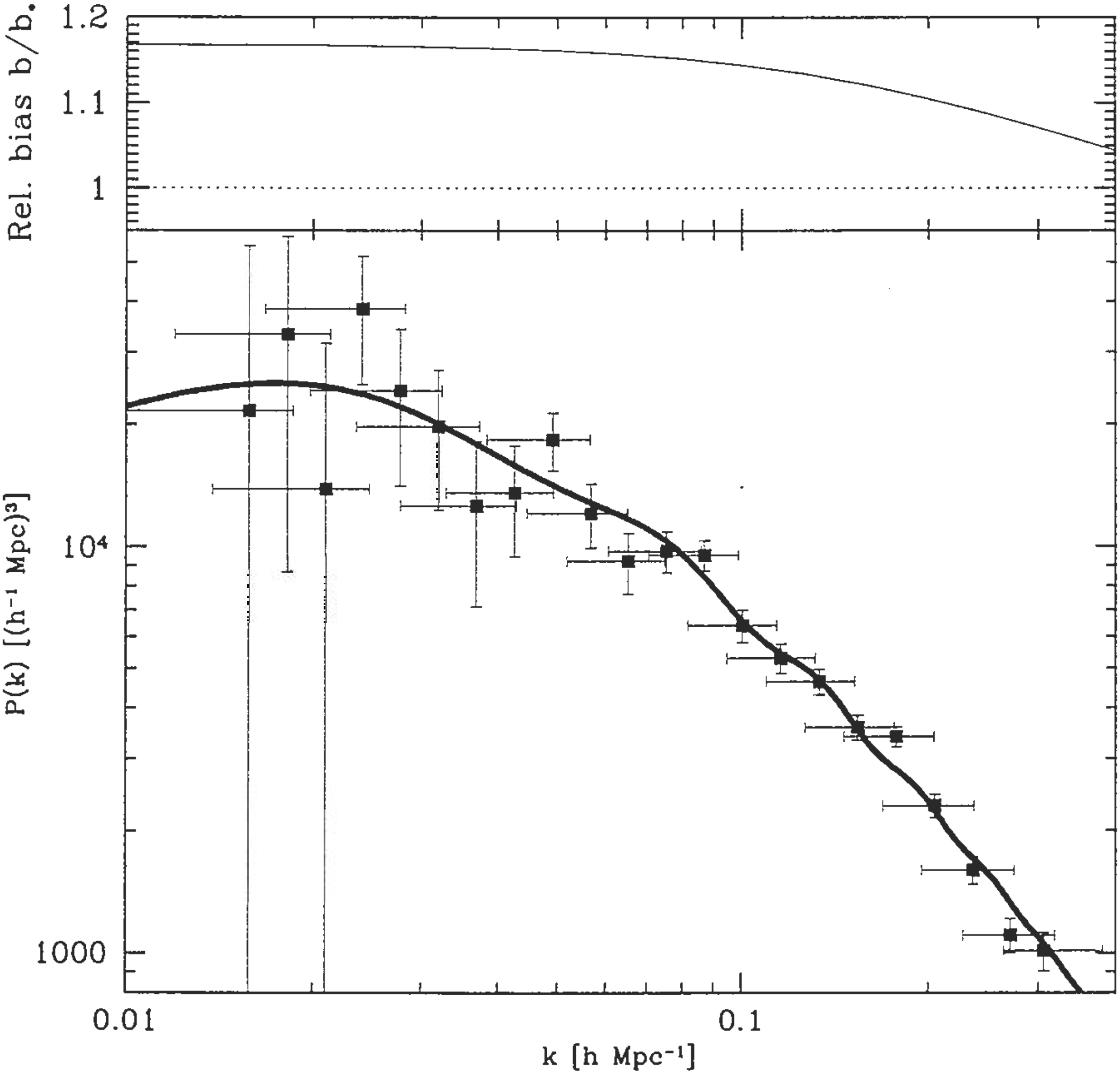


FIG. 22.— The decorrelated real-space galaxy-galaxy power spectrum using the modeling method is shown (bottom panel) for the baseline galaxy sample assuming $\beta = 0.5$ and $r = 1$. As discussed in the text, uncertainty in β and r contribute to an overall calibration uncertainty of order 4% which is not included in these error bars. To remove scale-dependent bias caused by luminosity-dependent clustering, the measurements have been divided by the square of the curve in the top panel, which shows the bias relative to L_* galaxies. This means that the points in the lower panel can be interpreted as the power spectrum of L_* galaxies. The solid curve (bottom) is the best fit linear ΛCDM model of Section 5.

D. E. Eisenstein et al., astro-ph/0501171

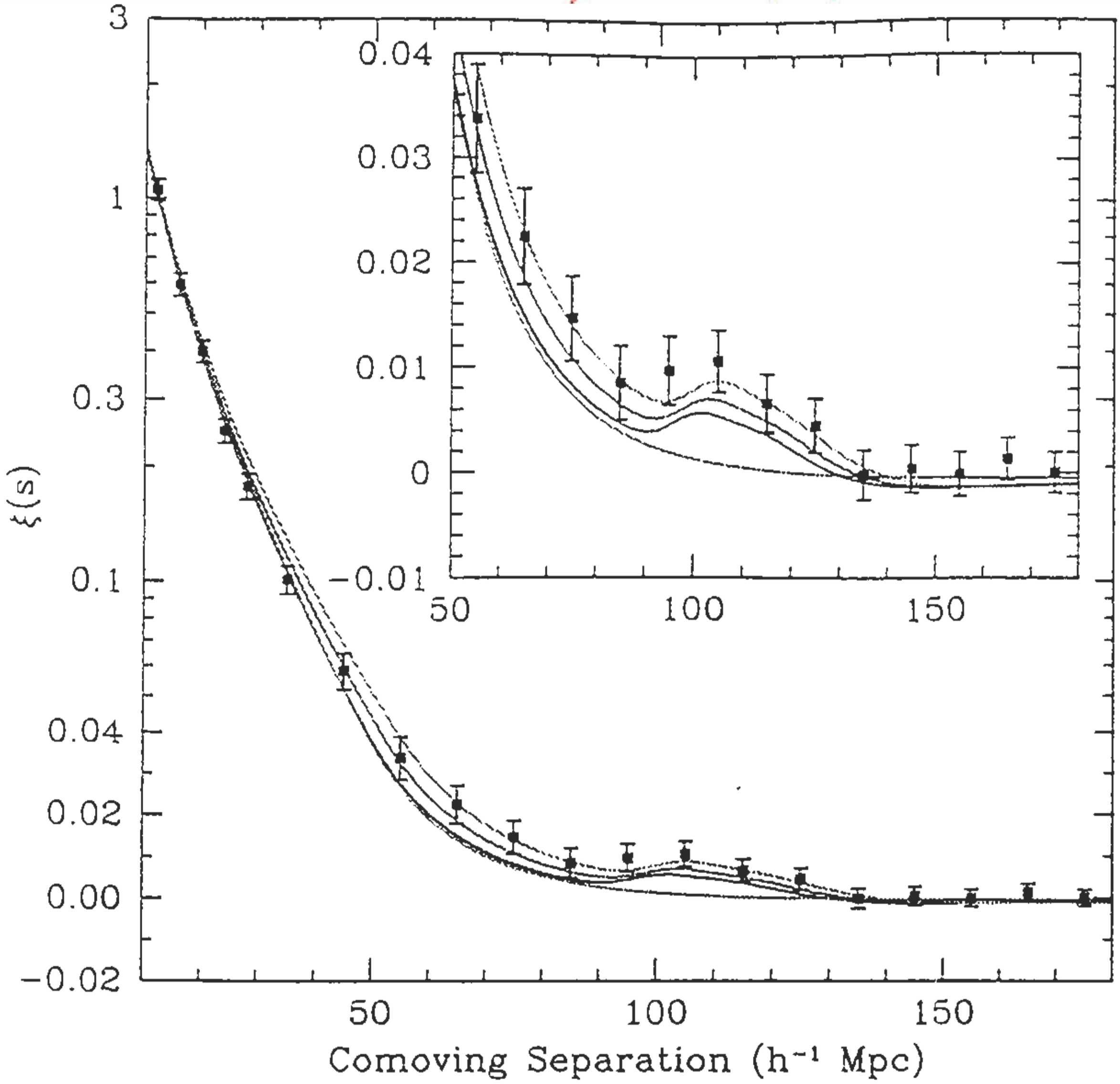


FIG. 2.— The large-scale redshift-space correlation function of the SDSS LRG sample. The error bars are from the diagonal elements of the mock-catalog covariance matrix; however, the points are correlated. Note that the vertical axis mixes logarithmic and linear scalings. The inset shows an expanded view with a linear vertical axis. The models are $\Omega_m h^2 = 0.12$ (top, green), 0.13 (red), and 0.14 (bottom with peak, blue), all with $\Omega_b h^2 = 0.024$ and $n = 0.98$ and with a mild non-linear prescription folded in. The magenta line shows a pure CDM model ($\Omega_m h^2 = 0.105$), which lacks the acoustic peak. It is interesting to note that although the data appears higher than the models, the covariance between the points is soft as regards overall shifts in $\xi(s)$. Subtracting 0.002 from $\xi(s)$ at all scales makes the plot look cosmetically perfect, but changes the best-fit χ^2 by only 1.3. The bump at $100h^{-1}$ Mpc scale, on the other hand, is statistically significant.

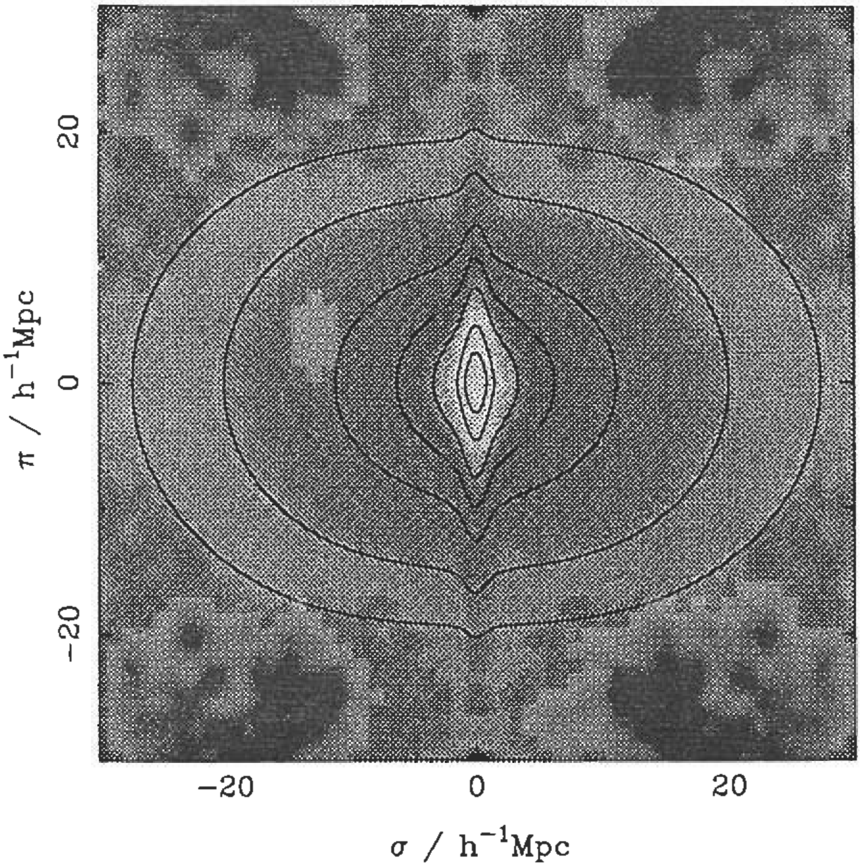


Figure 2 The redshift-space correlation function for the 2dFGRS, $\xi(\sigma, \pi)$, plotted as a function of transverse (σ) and radial (π) pair separation. The function was estimated by counting pairs in boxes of side $0.2\,h^{-1}$ Mpc (assuming an $\Omega = 1$ geometry), and then smoothing with a Gaussian of rms width $0.5\,h^{-1}$ Mpc. To illustrate deviations from circular symmetry, the data from the first quadrant are repeated with reflection in both axes. This plot clearly displays redshift distortions, with 'fingers of God' elongations at small scales and the coherent Kaiser flattening at large radii. The overplotted contours show model predictions with flattening parameter $\beta \equiv \Omega^{0.6}/b = 0.4$ and a pairwise dispersion of $\sigma_p = 400\,\mathrm{km\,s^{-1}}$. Contours are plotted at $\xi = 10, 5, 2, 1, 0.5, 0.2, 0.1$.

The model predictions assume that the redshift-space power spectrum (P_s) may be expressed as a product of the linear Kaiser distortion and a radial convolution[14]: $P_s(\mathbf{k}) = P_r(k)\,(1 + \beta\mu^2)^2\,(1 + k^2\sigma_p^2\mu^2/2H_0^2)^{-1}$, where $\mu = \hat{\mathbf{k}} \cdot \hat{\mathbf{r}}$, and σ_p is the rms pairwise dispersion of the random component of the galaxy velocity field. This model gives a very accurate fit to exact nonlinear simulations[15]. For the real-space power spectrum, $P_r(k)$, we take the estimate obtained by deprojecting the angular clustering in the APM survey[13,16]. This agrees very well with estimates that can be made directly from the 2dFGRS, as will be discussed elsewhere. We use this model only to estimate the scale dependence of the quadrupole-to-monopole ratio (although Fig. 2 shows that it does match the full $\xi(\sigma, \pi)$ data very well).

2. Power spectrum

(2) Approximately flat (Harrison 1970 - Zeldovich 1972)

$\overline{|n_s - 1|} < 0.1$ certainly

$n_s = 0.98 \pm 0.02$ (Seljak et al., 2004)

$$\langle \Phi_{in}^2 \rangle = \tilde{A}^2 \int \frac{dk}{k}$$ at the FRWMD stage

$$\tilde{A} = 2.9 \cdot 10^{-5} \cdot \left(\frac{A_{WMAP}}{0.9} \cdot e^{\tau - 0.17} \right)^{1/2}$$

3. Statistics

Gaussian

$\longleftrightarrow$

In more detail:

$$\overline{|n_s - 1|} \sim \frac{\text{few}}{N}$$

$$\overline{\left| \frac{dn_s}{d\ln k} \right|} \sim \overline{|n_s - 1|^2} \sim \frac{\text{few}}{N^2} \qquad N = 50 - 60$$

$$|r| \lesssim \frac{\text{few} \cdot 8}{N}$$

Robust predictions

How to change them?

1. Add more fields, non-slowly-rolling (at least, at some moments of time).
2. Assume "new physics" breaking the qualitative similarity between the two DS stages.

E.g., $O(1)\,\frac{H_{DS}}{M_{Pl}}$ relative correction to β_k is excluded

New (actually, very old) way of introducing the inflationary paradigm

Physical

In some period in the past, matter in the Universe was qualitatively the same as the main part of matter in the present Universe

Geometrical

Evolution of the Universe – transition between two maximally symmetric states (space-times, in particular)

Practical application: models of the same structure are used for description of the both dS stages

→ De Sitter ⇒ FRW ⇒ De Sitter →
(RD, MD)

"Quintessence – inflaton today";
k-essence ↔ k-inflation; $R+f(R)$; braneworlds; etc.

Final outcome of quantum inflationary cosmology

$$ds^2 = dt^2 - a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2 + h_{\alpha\beta}\, dx^\alpha dx^\beta\right)$$

$\lambda \gg ct$ $\quad h_{\alpha\beta} = \underbrace{h(\vec{r})\,\delta_\alpha^\beta}_{AP} + \underbrace{h_\alpha^{\beta(g)}}_{GW}$ $\qquad h_\alpha^{\alpha(g)} = 0$

$$h(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, e^{i\vec{k}\vec{r}} \frac{A}{k^{3/2}}\, a_k$$

$$h_\alpha^\beta(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \cdot e^{i\vec{k}\vec{r}} e_\alpha^\beta \frac{B}{k^{3/2}} c_{kj} \qquad j = 1, 2$$

$$e_\alpha^\alpha = 0, \quad e_\beta^\alpha(\vec{k})\, k^\beta = 0$$

$$\langle a_k a_{k'}^* \rangle = \delta^{(3)}(\vec{k} - \vec{k}')$$

$$\langle c_{kj} c_{k'j'}^* \rangle = \delta^{(3)}(\vec{k} - \vec{k}')\,\delta_{jj'}$$

a_k, c_{kj} – gaussian

(1979) $B^2 = 16\pi G H_k^2 = \frac{128\pi^2 G^2 V_k}{3}$; $\frac{k}{a(t_k)} = H(t_k)$

(1982) $A^2 = \frac{2H_k^4}{\dot\varphi_k^2} = \frac{H_k^4}{2\pi G|\dot H_k|} = \frac{18\left(\frac{8\pi G}{3} V_k\right)^3}{V_k'^2}$

weak dependence on k

(1981 – for geometrical inflation)

$$\boxed{\begin{aligned} \dot\varphi &\approx -\frac{V'}{3H} \\ H^2 &\approx \frac{8\pi G V}{3} \end{aligned}}$$

More generally

$$h(\vec{r}) = 2\,\delta\left(\ln \frac{a(t_f(\vec{r}))}{a(t_{in})}\right)$$

Connection to the longitudinal gauge

$$ds^2 = (1 + 2\Phi)\,dt^2 - (1 - 2\Psi)(dx^2 + dy^2 + dz^2)$$

$$\Phi = \Psi = -\frac{h}{2}\left(1 - \frac{H}{a}\int_0^t a\,dt\right) \qquad \left(= -\frac{3}{10} h(\vec{r})\right)$$

if $a(t) \propto t^{2/3}$

Larger local amount of inflation $\rightarrow$ $\underline{\Phi < 0}$

$\frac{\Delta T}{T} < 0$

Simple derivation (A.S., 1982)

$H = H_0$ during inflation
(for simplicity)

Beginning of inflation $\quad t = 0$

End of inflation $\quad t = t_0(\vec{r})$

Let $H \simeq \text{const}$ during inflation

$$ds^2 = dt^2 - a_0^2 e^{2Ht} dl^2 \equiv$$

$$\equiv dt^2 - a_0^2 e^{2Ht_0(\vec{r})} e^{2H(t - t_0(\vec{r}))} dl^2 \rightarrow$$

Quantum perturbations:

results in $\varphi = \varphi(t - t_0(\vec{r}))$
$a = a(t - t_0(\vec{r}))$

$$\rightarrow dt^2 - e^{2Ht_0(\vec{r})} a^2(t - t_0(\vec{r})) dl^2 \rightarrow$$

exact homogeneous solution

$$\rightarrow dt^2 - a_1^2 e^{2Ht_0(\vec{r})} (t - t_0(\vec{r})) dl^2 \rightarrow$$

at the radiation-dominated stage

$$\rightarrow dt^2 - a_1^2 e^{2Ht_0(\vec{r})} t \, dl^2$$

Present observational situation:

no positive results beyond SCM,

but beginning to exclude some inflationary models

E.g., $V(\varphi) \propto \varphi^4$ is on verge ($\sim 3\sigma$) without Ly-α and excluded with them.

However, the earliest and simplest models are still alive and o.k.

1) $V(\varphi) \propto \varphi^2$ $\quad n_s = 1 - \frac{2}{N} = 0.96$, $r = \frac{8}{N} = 0.16$

(analytical solution for the background model known since 1978, used as an inflationary model in 1983)

2) $R + R^2$ $\quad n_s = 0.96$, $r = \frac{12}{N^2} \approx 3 \cdot 10^{-3}$

(1980)

3) $V(\varphi) = V_0 - \frac{\lambda \varphi^4}{4}$ $\quad n_s = 1 - \frac{3}{N} = 0.94$, $r \ll 1$

"new inflation" (1982)

Higher-order corrections to the slow-roll approximation

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{1}{4\pi G}\left(\frac{H'}{H}\right)^2 \qquad ' = \frac{d}{d\varphi}$$

$$\eta \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}} = \frac{1}{4\pi G}\frac{H''}{H}$$

$$\xi = \frac{1}{4\pi G}\left(\frac{H' H'''}{H^2}\right)^{1/2}$$

and so on ...

$\epsilon, |\eta|, \xi \ll 1 \longrightarrow$ slow-roll

Relation to scales of inhomogeneities

$$\frac{d \ln k}{d\varphi} = \frac{1}{4\pi G}\frac{H}{H'}(\epsilon - 1)$$

Euler's constant

$$n_S - 1 = -4\epsilon + 2\eta - [8(1+C)\epsilon^2 - (6+10C)\epsilon\eta + 2C\xi^2] + O(\epsilon^3, \eta^3, \xi^3)$$

$$n_T = -2\epsilon[1 + (3+2C)\epsilon - 2(1+C)\eta] + \ldots$$

ϵ^2-correction $\rightarrow$ 1993 (Lyth & Stewart)
ϵ^3-correction $\rightarrow$ 2001 (Stewart & Gong)

Exact solution of the equations

$$3H^2 = \frac{8\pi G}{3}\left(\frac{\dot\varphi^2}{2} + V(\varphi)\right)$$

$$\ddot\varphi + 3H\dot\varphi + \frac{dV}{d\varphi} = 0$$

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2}\right)u_k = 0$$

$$u = Qa, \quad H = \frac{\dot a}{a}, \quad z = \frac{a\dot\varphi}{H}, \quad \eta = \int \frac{dt}{a(t)}$$

with the initial condition

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}}$$

for $k\eta \to -\infty$.

set of potentials producing the same perturbation spectrum.
In particular, the problem of accuracy of the slow-roll approximation prediction for $P_0(k)$ (including higher order corrections) has been intensively and critically studied recently using different methods: [14], [15] (the uniform approximation), [16] (the improved WKB-approximation) and others.

By an exact solution I mean a solution of the following system of equations for a spatially-flat Friedmann-Robertson-Walker (FRW) background with a scale factor $a(t)$ and scalar (adiabatic) perturbations described by the Mukhanov variable $Q \equiv u/a$:

$$H^2 = \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) , \tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 , \tag{2}$$

$$\frac{d^2u_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2z}{d\eta^2}\right) u_k = 0 , \tag{3}$$

obtained without any approximations. Here

$$H = \frac{\dot{a}}{a} , \quad z = \frac{a\dot{\phi}}{H} , \quad \eta = \int \frac{dt}{a(t)} , \tag{4}$$

dot means the derivative with respect to t, $u_k(\eta)\exp(i\mathbf{kr})$ is the wave function of a Fourier mode of the quantum field u (the c-number multiplying the Fock annihilation operator $\hat{a}_{\mathbf{k}}$), and $c = \hbar = 1$ is put throughout the paper. The variable Q [17] is equal to $\delta\phi_L + \frac{\dot{\phi}}{H}\Phi$ in the longitudinal gauge (Φ is the quasi-Newtonian gravitational potential), or to $\delta\phi_S - \frac{\dot{\phi}}{6H}(\mu+\lambda)$ in the synchronous gauge (μ and λ are the Lifshits variables). The normalized initial condition for u_k corresponding to the adiabatic vacuum at $t \to -\infty$ ($\eta \to -\infty$) is

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}} . \tag{5}$$

At late times during an inflationary stage in the super-horizon regime ($k \ll aH$, $\eta \to 0$),

$$\frac{u_k}{z} = \frac{HQ_k}{\dot{\phi}} \to const = \zeta(k) \tag{6}$$

($\zeta = -h/2$ in the notation of [4]).

Then the initial spectrum of adiabatic perturbations for a post-inflationary cosmology in the super-horizon regime is (assuming the absence of non-diagonal pressure components):

$$< \Phi^2 >= \left(1 - \frac{H}{a}\int_0^t a\, dt\right)^2 < \zeta^2 >= \left(1 - \frac{H}{a}\int_0^t a\, dt\right)^2 \int P_0(k)\,\frac{dk}{k} , \quad P_0(k) = \frac{k^3\zeta^2(k)}{2\pi^2} . \tag{7}$$

Here $t = 0$ corresponds to the end of inflation. For historical reasons, the slope n_S of the spectrum is defined with respect to density perturbations in the non-relativistic dark matter + baryon component at the present time, $(\delta\rho)_k = -k^2\Phi_k/4\pi Ga^2$ before integration over d^3k.

So, $n_S = 1 + \frac{d\ln P_0(k)}{d\ln k}$. Finally, using the equation $\dot{H} = -4\pi G\dot{\phi}^2$ that follows form Eqs. (2) and (3), Eq. (2) can be recast in the Hamilton-Jacobi form [18]:

$$H^2(\phi) - \frac{H'^2(\phi)}{12\pi G} = \frac{8\pi G}{3}V(\phi) \ , \tag{8}$$

where the prime denotes the derivative with respect to ϕ.

Exact solutions of the inverse problem of reconstruction of $V(\phi)$ given $P_0(k)$ are known for the following two cases only, if not speaking about solutions describing universes collapsing towards a singularity.

1) A power-law perturbation spectrum with the slope $n_S = const < 1$ [19]. Then

$$V(\phi) \propto H^2(\phi) \propto \exp\left(\pm\sqrt{\frac{16\pi G}{q}}\phi\right) \ , \quad a(t) \propto t^q \ , \quad q = \frac{3-n_S}{1-n_S} > 1 \ . \tag{9}$$

This is just the power-law inflation. Considered as a function of $\phi(t)$, H is related to $V(\phi)$ through Eq. (8). Note, however, that this is not the only potential producing the $n_S = const < 1$ spectrum.

2) The case when no perturbations are generated at all (no real created quanta of the inflaton field) [20]:

$$H(\phi) = H_1 \exp(2\pi G\phi^2) \ , \quad V(\phi) = \frac{3H_1^2}{8\pi G}\left(1 - \frac{4\pi G\phi^2}{3}\right)\exp(4\pi G\phi^2) \ . \tag{10}$$

In literature, this case is sometimes incorrectly referred as the potential generating the $n_S = 3$ perturbation spectrum. However, one should not forget that generated perturbations are quantum (even quantum-gravitational) and require renormalization. After subtraction of the vacuum energy $\omega(t)/2 = k/2a(t)$ of each mode, no created fluctuations remain in this case. Moreover, a number of real inflaton quanta generated in each perturbation mode k should be large, because in the opposite case they may not be interpreted as classical perturbations after the end of inflation (see [21] for a more detailed discussion of this point).

Strictly speaking, there is no exit from inflation for the potential (9), and the potential (10) does not admit a low curvature regime at all. However, in the former case $V(\phi)$ can be deformed such that it reaches zero at a sufficiently large value of ϕ. This will result in a very small change of the perturbation spectrum at present scales of interest that may be safely neglected. Sometimes, the case of a parabolic potential near its maximum $V(\phi) = V_0 - \frac{m^2\phi^2}{2}$ is mentioned as an exactly soluble case. However, it is not such the one in our terminology since in this case $H(\phi)$ is approximated by the constant value $H_0 = \sqrt{8\pi G V_0/3}$.

In this paper, a family of exact solutions for the case $n_S = 1$ is constructed. It is just the initial spectrum proposed by Harrison and Zeldovich [22], after all, for beauty reasons. Note that it satisfies the most recent CMB data [23, 24]. Let us first consider what follows for this case from the slow-roll approximation. Then, the leading term in the power spectrum reads

$$k^3\zeta^2(k) \propto \left(\frac{V^3}{V'^2}\right)_{t=t_k} \ , \tag{11}$$

where t_k is the moment when $k = aH$. It is clear that, to get $n_S = 1$, $V^{3/2}/V'$ should not depend on ϕ. Therefore, $V(\phi) \propto \phi^{-2}$. Note that this solution of the reconstruction

"Semi-exact" solutions

$$H(\varphi) \simeq H_0 = \text{const}$$

1. Inflation near a maximum of $V(\varphi)$

$$V = V_0 - \frac{m^2\varphi^2}{2}\,, \quad H_0^2 = \frac{8\pi G V_0}{3}$$

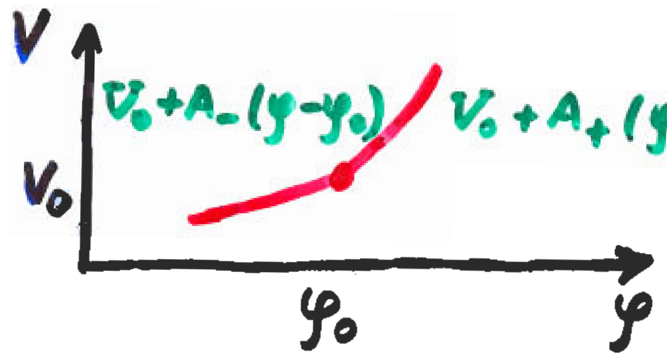
$$n_S = 4 - \sqrt{9 + 4\frac{m^2}{H_0^2}} < 1 \qquad \left(\begin{matrix} >1, \\ \text{if } m^2 < 0 \end{matrix}\right)$$

2. Inflation near a jump in $V'(\varphi)$

2. $[V]=0,\ [V']\neq 0$ at $\varphi=\varphi_0$

(and $\ll \sqrt{48\pi G V_0}$)

A.A.S., JETP Lett. 55 (1992) 489



Slow-roll approximation does not work.

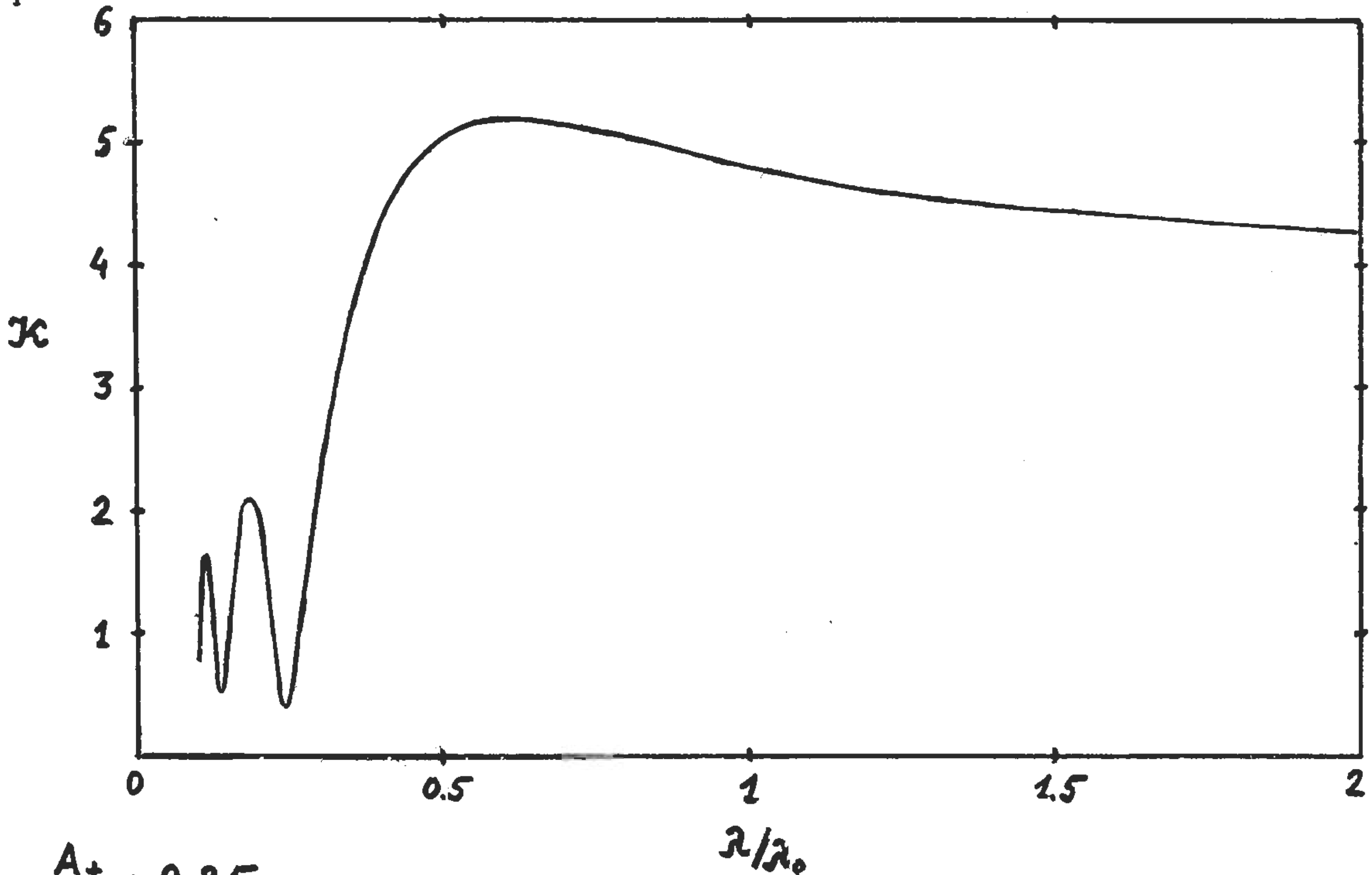
Universal local spectrum:

$$P_0(k)=\left[1-6\,\frac{A_+-A_-}{A_+}\,\frac{k_0^2}{k^2}\left(-\cos\frac{2k}{k_0}+\frac{k_0^2-k^2}{2kk_0}\sin\frac{2k}{k_0}\right)+\frac{9(A_+-A_-)^2}{A_+^2}\,\frac{k_0^4}{k^4}\cdot\left(\frac{(k^2+k_0^2)^2}{2k^2k_0^2}-\frac{k^2+k_0^2}{kk_0}\sin\frac{2k}{k_0}-\frac{k_0^4-k^4}{2k^2k_0^2}\cos\frac{2k}{k_0}\right)\right]$$

$$\cdot\, P_{reg}(k)$$

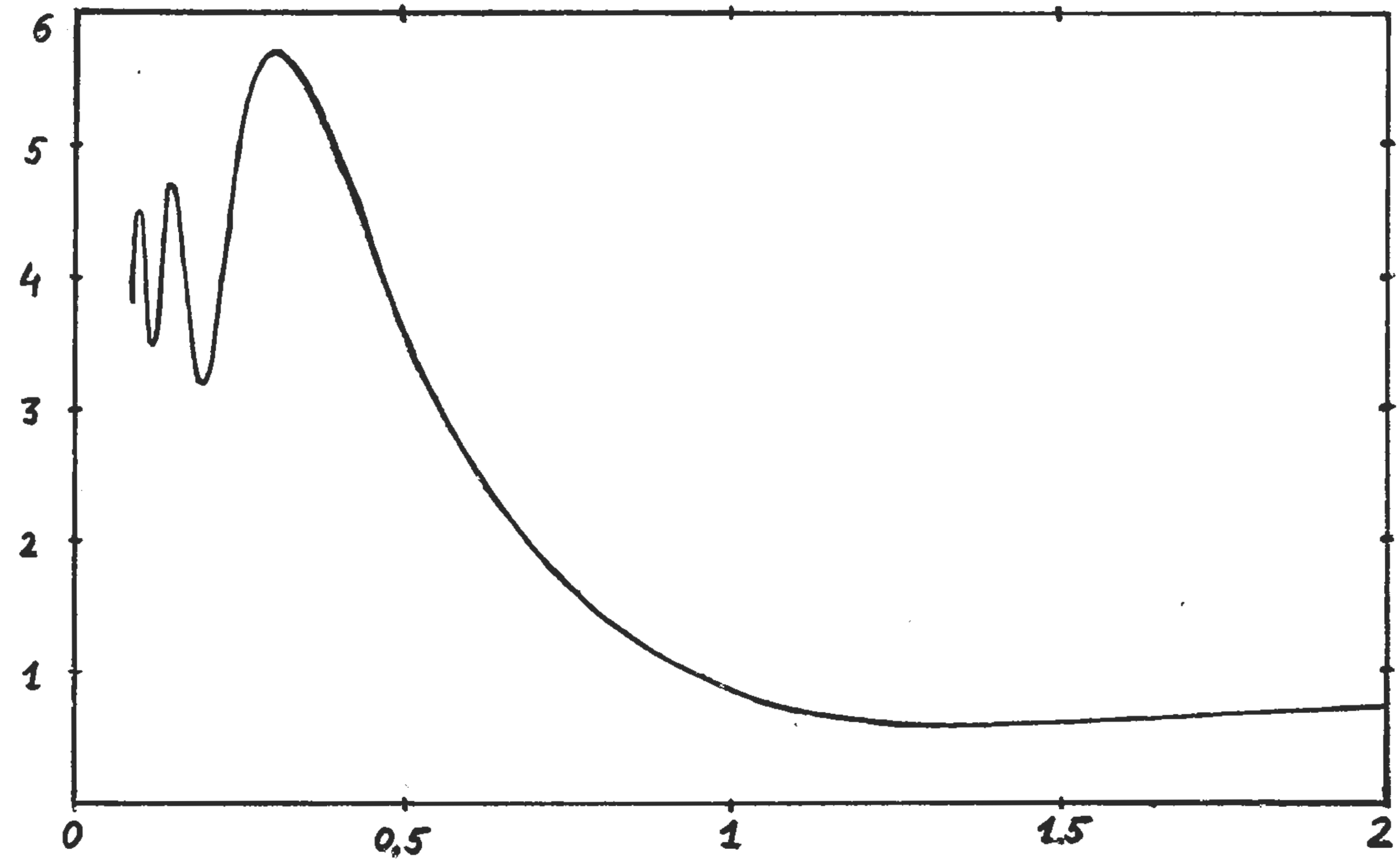
$$k_0=a(t_0)H(t_0)$$

$$P_0(k) = P_{reg}(k) \cdot \mathcal{K}^2$$



$$\frac{A_+}{A_-} = 0.25$$

$$P_0(k) = P_{reg}(k) \cdot \mathcal{K}^2$$



$\frac{A_+}{A_-} = 4$

problem is unique for a given amplitude of the flat spectrum. This kind of inflation was dubbed intermediate inflation in [25] (see also [26]). Its scale factor behaviour is $a(t) \propto \exp\left(const \cdot t^{2/3}\right)$. Once more, it does not have an exit from inflation, so it should be modified at large ϕ. A next order slow-roll correction to this potential was considered in [27].

To obtain an exact solution for $H(\phi)$ and $V(\phi)$ in the case $n_S = 1$, note first that, for

$$\frac{1}{z}\frac{d^2 z}{d\eta^2} = \frac{2}{\eta^2}\,, \tag{12}$$

Eq. (3) reduces to the equation for a massless scalar field in the de Sitter background and has the solution

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \tag{13}$$

satisfying the initial condition (5). Let us write the general solution of Eq. (12) in the form

$$z = \frac{B}{|\eta|}\left(1 + \frac{|\eta|^3}{\eta_0^3}\right), \quad \eta < 0\,, \tag{14}$$

where A, η_0 are constants. The limiting case $\eta_0 \to 0$, when the first term in brackets may be neglected, is not interesting because it corresponds to a collapsing universe (however, it is "dual" to the case $\eta_0 \to \infty$ considered below). The power spectrum of the growing perturbation mode is $P_0(k) = 1/4\pi^2 B^2$ and does not depend on η_0 (η_0 appears in the amplitude of the decaying mode only and makes it non-scale-free). Thus, we have got the exactly flat spectrum. Present observational CMB data [24] fix the quantity B with $\approx 10\%$ accuracy:

$$\frac{1}{2\pi B} = 4.8 \cdot 10^{-5}\left(\frac{A}{0.9}\exp(\tau - 0.17)\right)^{1/2}, \tag{15}$$

where A is the quantity introduced in [24] and τ is the optical length after recombination. In this notation, $A = 0.9$ corresponds to the value $A = 4.3 \cdot 10^{-4}$ of the other quantity A introduced in [28] to characterize an amplitude of initial perturbations (and conjectured to lie in the range $(3 - 10) \cdot 10^{-4}$ in that paper).

Since the aim of this paper is to find *some* exact solution, I will not investigate if there exist other forms of z leading to the $n_S = 1$ spectrum, too. The absence of other solutions for z would immediately follow from scaling arguments if we assume that $u_k \propto k^{-1/2} f(k\eta)$ for *all* η. However, the latter assumption might not be necessary. Moreover, I will consider only one particular case of Eq. (14) corresponding to the limit $\eta_0 \to \infty$.

So, let $z = -B/\eta$. Let us express all quantities of interest as functions of ϕ:

$$\begin{aligned} t &= -4\pi G \int \frac{d\phi}{H'}\,, \quad \ln a = \int H(t)\,dt = -4\pi G \int \frac{H}{H'}\,d\phi\,, \\ \eta &= \int \frac{dt}{a(t)} = -4\pi G \int \frac{d\phi}{H'} \exp\left(4\pi G \int \frac{H}{H'}\,d\phi\right), \\ z &= \frac{a\dot\phi}{H} = -\frac{H'}{4\pi G H} \exp\left(-4\pi G \int \frac{H}{H'}\,d\phi\right). \end{aligned} \tag{16}$$

Equating the last line in Eq. (16) to $-B/\eta$, we get the following equation:

$$\int P(\phi)\,d\phi = -BHP\ ,\quad P \equiv \frac{4\pi G}{H'}\exp\left(4\pi G\int\frac{H}{H'}\,d\phi\right)\ . \tag{17}$$

After differentiation, Eq. (17) reduces to $P = -B(HP' + H'P)$, or

$$\frac{4\pi GH^2}{H'} - \frac{HH''}{H'} + H' + \frac{1}{B} = 0\ . \tag{18}$$

Let us introduce dimensionless variables

$$x = \sqrt{4\pi G}\phi\ ,\quad y = B\sqrt{4\pi G}H\ ,\quad v(x) = \frac{32\pi^2G^2B^2}{3}V(\phi)\ . \tag{19}$$

Then, from (8), $v = y^2 - (1/3)(dy/dx)^2$. For these variables, Eq. (18) reads:

$$y\,\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + y^2\ . \tag{20}$$

After dividing by y^2, the last equation can be integrated to $dy/dx = xy - 1$ (an integration constant is excluded by shifting x, i.e., ϕ). Therefore,

$$y = e^{x^2/2}\left(\int_x^\infty e^{-\tilde{x}^2/2}\,d\tilde{x} + C\right)\ , \tag{21}$$

where C is another integration constant. This just yields us a one-parameter family of solutions having $n_S = 1$. The so-called slow-roll parameters for this solution:

$$\epsilon(\phi) \equiv \frac{1}{4\pi G}\frac{H'^2}{H^2} = \left(\frac{1}{y} - x\right)^2,$$
$$\tilde{\eta}(\phi) \equiv \frac{1}{4\pi G}\frac{H''}{H} = \frac{1}{y}\frac{d^2y}{dx^2} = x^2 - \frac{x}{y} + 1\ . \tag{22}$$

The partial solution with $C = 0$ has an infinite inflationary stage which is just described by the slow-roll approximation for $x \gg 1$. Its graph is plotted in Fig.1. Its large-x expansion is

$$y = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \dots,\quad v = \frac{1}{x^2} - \frac{7}{3x^4} + \frac{9}{x^6} - \dots \tag{23}$$

It is straightforward to check that it leads to $n_S = 1$ (as it should be) for the first [8] and second [9] order corrections to the slow-roll approximation. However, these corrections miss the whole 1-parametric family with $C \neq 0$ completely.

For $x < 0$, the solution with $C = 0$ has rather peculiar behaviour: the potential $v(x)$ reaches the maximum value $v_{max} \approx 7.252$ at $x \approx -1.326$, becomes zero at $x \approx -1.618$ and then going to $-\infty$ at $x \to -\infty$ (however, such effective potentials are considered in string inspired models now). In the latter limit, $y \to \infty$, so we get an initial curvature singularity at a finite proper time $t_0 < 0$. If $t = 0$ is the moment when $x = 0$ ($v(0) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$) and the inflationary stage begins, then $|t_0| \sim H^{-1}(0) \sim BG^{1/2}$. The scale factor reaches zero very slowly: $a(t) \propto |\ln(t - t_0)|^{-1/2}$ for $t \to t_0$. Still the Riemann tensor is not twice integrable for

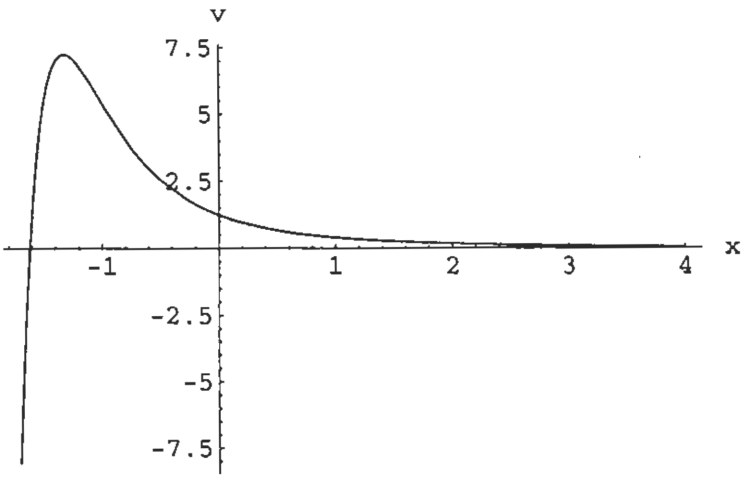


Figure 1: The dimensionless potential $v(x)$ for the $C = 0$ case.

$t \to t_0$, so this singularity is a strong one. The same refers to all initially expanding ($y > 0$) solutions with $C \neq 0$ and $C > -\sqrt{2\pi}$ – they all begin from such a singularity.

By taking $C < 0$ and very small, it becomes possible to construct a solution with a long but finite inflationary stage. Namely, if $C = -\sqrt{3}x_1^{-2}\exp(-x_1^2/2)$ with $x_1 \gg 1$, then $v(x)$ becomes zero at $x = x_1$ (y still remains $\sim x_1^{-1}$). In this case inflation ends (ϵ, $|\tilde{\eta}| \sim 1$) at $x = x_1 - O(x_1^{-1})$. The total number of e-folds is $N_{tot} = 2\pi G\phi_1^2 = x_1^2/2$. Thus, $C \sim \exp(-N_{tot})$ that is in agreement with the general principle that terms not caught by an arbitrary order of a WKB-type expansion are exponentially small. For $x \geq x_1$, one may put $v \equiv 0$. Then the kinetic dominated phase $a(t) \propto t^{1/3}$ follows the inflationary stage. Or, we may assume that v has a local minimum $v = \frac{1}{2}\mu^2(x - x_1)^2$ around this point. It results in oscillations in ϕ and the matter-dominated post-inflationary stage $a(t) \propto t^{2/3}$.

Finally, note that the spectrum of gravitational waves (GW) is not flat for this model: for $1 \ll x \ll x_1$, the tensor-scalar ratio and the slope of the GW initial power spectrum $r = -8n_T = 16/x^2 = 8/N$ where N is the number of e-folds from the *beginning* of inflation. The present upper observational bound $r < 0.36$ [29] requires $N > 22$ for the comoving scale crossing the Hubble radius at present. So, N_{tot} should exceed ~ 70 in this model.

The research was partially supported by the Russian Foundation for Fundamental Research, grant No. 05-02-17450, by the Research Programme "Elementary Particles" of the Russian Academy of Sciences and by the scientific school grant No. 2338.2003.2 of the Russian Ministry of Education and Science.

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EXPECTED FUTURE DISCOVERIES

New effects, new (but small) fundamental constants.

1. $n_s = 1 + f(k)$, $\overline{|n_s - 1|} \sim \frac{\text{few}}{N}$

$N = 50-60$

$n_s \equiv 1$ is possible for a special class of $V(\varphi)$ only.

In the slow-roll approximation: $V(\varphi) \propto \varphi^{-2}$

Exact: 1 parametric family explicitly

$$y \equiv \frac{\sqrt{4\pi G}\, H(\varphi)}{B} = \exp\left(\frac{x^2}{2}\right)\left(\int_x^\infty \exp\left(-\frac{\tilde{x}^2}{2}\right) d\tilde{x} + C\right);$$

$$x = \sqrt{4\pi G}\,\varphi \; ; \quad \langle k^3 \zeta^2(\vec{k}) \rangle = \frac{B^2}{2} \; ;$$

$$V(\varphi) = \frac{3B^2}{32\pi^2 G^2}\left(y^2(x) - \frac{1}{3} y'^2(x)\right).$$

2 parametric family exists

(A.S., JETP Lett. 82, 169 (2005)
astro-ph/0507193)

Lessons and subtleties

1. $P(k)$ is determined by $H(\varphi)$, not $V(\varphi)$ only. $H(\varphi)$ determines both $V(\varphi)$ and $\dot{\varphi}$. For a given $V(\varphi)$, there may be solutions with different $H(\varphi)$.

2. $a(t)$ is singular in the past, before inflation. As a result, a slightly different problem has been solved exactly: not with $u_k \to \frac{e^{-ik\eta}}{\sqrt{2k}}$ for $k\eta \to -\infty$, but with $u_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)$ for all admissible η.

3. The slow-roll expansion in an asymptotic one only (i.e., not convergent).

4. A decaying mode of φ (the term $\propto \eta_0^{-3}$ in Eq. (14)) does not affect $P(k)$ directly. However, an effect through a change of $H(\varphi)$ for a given $V(\varphi)$ is possible.

Another exactly integrable case:

$$\frac{k_g^2}{k_s^2} \equiv \frac{B^2(k)}{A^2(k)} = \text{const}$$

Known particular case: the power-law inflation

$$\frac{z''}{z} = \frac{a''}{a} \qquad z = \frac{a\dot{\varphi}}{H}$$

$$\frac{\dot{\varphi}}{H} = C_1 + C_2 \int_{\infty}^{t} \frac{dt}{a^3}$$

General solution:

$$x = \sqrt{4\pi G}\,\varphi \ , \quad y = \sqrt{4\pi G}\, H$$

$$x = \int \frac{w\,dw}{3w - \frac{w^3}{3} + C}$$

$$\ln y = \int w\,dx = \int \frac{dw \cdot w^2}{3w - \frac{w^3}{3} + C}$$

$$t = -\sqrt{4\pi G} \int \frac{dw}{3w - \frac{w^3}{3} + C} \exp\left(-\int \frac{dw \cdot w^2}{3w - \frac{w^3}{3} + C}\right)$$

$$a = a_0 \exp\left(-\int \frac{dw}{3w - \frac{w^3}{3} + C}\right)$$

2 physical arbitrary parameters

2. Running: expected $\left|\frac{dn_s}{d\ln k}\right| \sim |n_s - 1|^2 \sim 10^{-3}$

Recent results including Ly-α data (Viel et al. (2004), Seljak et al. (2004)): no running at the $\sim 10^{-2}$ level

3. Primordial GW background

$r(k)$, $n_T(k)$

(first observational prediction of inflationary models: A.S., 1979)

For many models: $r = -8 n_T$ but not always.

$$\boxed{r \sim \frac{8}{N} \lesssim 0.1 \text{ a non-trivial prediction of inflation!}}$$

Best present upper limit: $r < 0.36$ (95%)
(Seljak et al. (2004))

4. Non-Gaussianeity

Expected: small $f \sim 1$

$$R = R_{lin} + R^2_{lin} + \dots$$
$$\Phi = \Phi_{lin} - \frac{5}{3}\Phi^2_{lin} + \dots$$

Current upper limit $|f| \lesssim 100$

5. Isocurvature modes

Possible, but certainly non-dominant

Upper bounds only, no definite prediction of smallness

6. Local features in $P_{in}(k)$.

May signalize fast phase transitions during inflation in other fields than inflaton.

Many models (including double and multiple inflation).

Some small features are seen in SDSS and 2dF $P(k)$ (e.g., bump at $k=0.05h$ Mpc^{-1} and well at $k=0.035h$ Mpc^{-1} proposed by Einasto et al. (1996)) but not enough statistically significant.

7. Local features in C_ℓ.

E.g. at $\ell \simeq 50$ and $\ell \simeq 200$ (Archeops, WMAP)

8. The low $\ell=2$ (and partly $\ell=3$) problem

From observations:

a) Statistical significance unclear, contamination by Galaxy

b) Always may be attributed to an accidence

c) Statistical isotropy unclear

From theory:

a) A cut-off of $P_0(k)$ for $k \to 0$ does not give too much (no suppression below the Sachs-Wolfe plateau)

b) Non-trivial spatial topology may help better, but predicts many effects for larger ℓ - - not seen at present, but ...

Too few information. Things worth to be done:

a) disentanglement of the ISW contribution to $\ell = 2, 3$;

b) $Q(z)$ using the SZ effect on clusters.

As a whole: not a critical problem at present, but should be kept in mind.

POST-INFLATIONARY GENERATION OF THE ADIABATIC MODE

1. Curvaton (Enqvist & Sloth; Lyth; Moroi)
2. Modulated fluctuations (Kofman; Dvali, Gruzinov, Zaldarriaga ....)

Common properties with standard multiple inflation:

1. Assume an inflationary stage
2. Use the same mechanism of light scalar field fluctuations generation during inflation

New element:

These scalar field fluctuations become imprinted into scalar adiabatic metric perturbations after the end of inflation

After that, no isocurvature modes remain.

Unifying formula
(Polarski & A.S., forthcoming)

$$\zeta(\vec{r}) = -\Delta N\Big|_{\text{1st horizon crossing}}^{t_{rec}\ \text{(or 2nd horizon crossing)}} = -\frac{\delta N}{\delta \varphi_a}\,\delta\varphi_a(\vec{r})$$

Calculated using a background solution

at 1st horizon crossing

Assuming no isocurvature modes at recombination

For the adiabatic mode – only total number of e-folds is important!

Low ℓ: $\frac{\Delta T}{T} = \frac{1}{3}\Phi = -\frac{3}{5}\Delta N$

Examples.

1. Massive curvaton σ.

$$N = N_0 + \frac{1}{2}\ln\frac{t_{eq}}{t_{reh}} + \frac{2}{3}\ln\frac{t_1}{t_{eq}}$$

$t_{eq} \propto \sigma^{-4}$

$$\zeta = -\left(\frac{1}{2}-\frac{2}{3}\right)\frac{\delta t_{eq}}{t_{eq}} = -\frac{2}{3}\frac{\delta\sigma}{\sigma}$$

2. Modulated decay of an inflaton

$$N = N_0 + \frac{2}{3}\ln\frac{1}{\Gamma t_f} + \frac{1}{2}\ln \Gamma t_{eq}$$

$$\zeta = \frac{1}{6}\frac{\delta\Gamma}{\Gamma} = \frac{1}{6}\frac{\delta\ln\Gamma}{\delta\chi}\delta\chi$$

CONCLUSIONS

1. All zero-order predictions confirmed.
2. The scalar primordial perturbation spectrum is a powerful, but degenerate tool for investigation of the very early Universe
3. New level of accuracy ~ 1% is needed for a number of expected and unexpected discoveries
4. The most certain expected effect:

   $$n_s - 1 = f(k), \quad \left|\frac{df}{d\ln k}\right| \ll 1$$

5. Further follow:
   a) GW, r
   b) "local features"
   c) some completely new physics
6. Remote task:

   direct reconstruction of $P_0(k)$ and $H(\varphi)$ from observational data in a model-independent way

7. Another general consequence of the approximately flat initial spectrum:

rich physics of pregalactic objects ("Population III")

Strongly depends on both $n_s - 1$ and DM properties

For $n_s = 1$ and very heavy DM particles

$$\left\langle k^3 \left(\frac{\delta\rho}{\rho}\right)^2_k \right\rangle \propto \ln^2 k \qquad k \to \infty$$

$$\sigma(10^9 M_\odot) = 1 \quad \text{at} \quad z \sim 10$$
$$\sigma(10^6 M_\odot) = 1 \quad \text{at} \quad z \sim 20$$

8. Quantum-to-classical transition for perturbations

Much better and much more important object for "gedunken" experiments that the Schrödinger's cat!

Something non-trivial may be in effects $\propto \ell^2 \sim 10^{-10}$.