Coherent transport in frustrated Josephson-junction rhombi chain with quenched disorder

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Plan

- Regular chain with no disorder: $4e$- versus $2e$- supercurrent
- Effective Hamiltonian in presence of random stray charges
- Chain with GIVEN disorder
- Statistical description of the chain
- Conclusions

Regular chain of rhombi

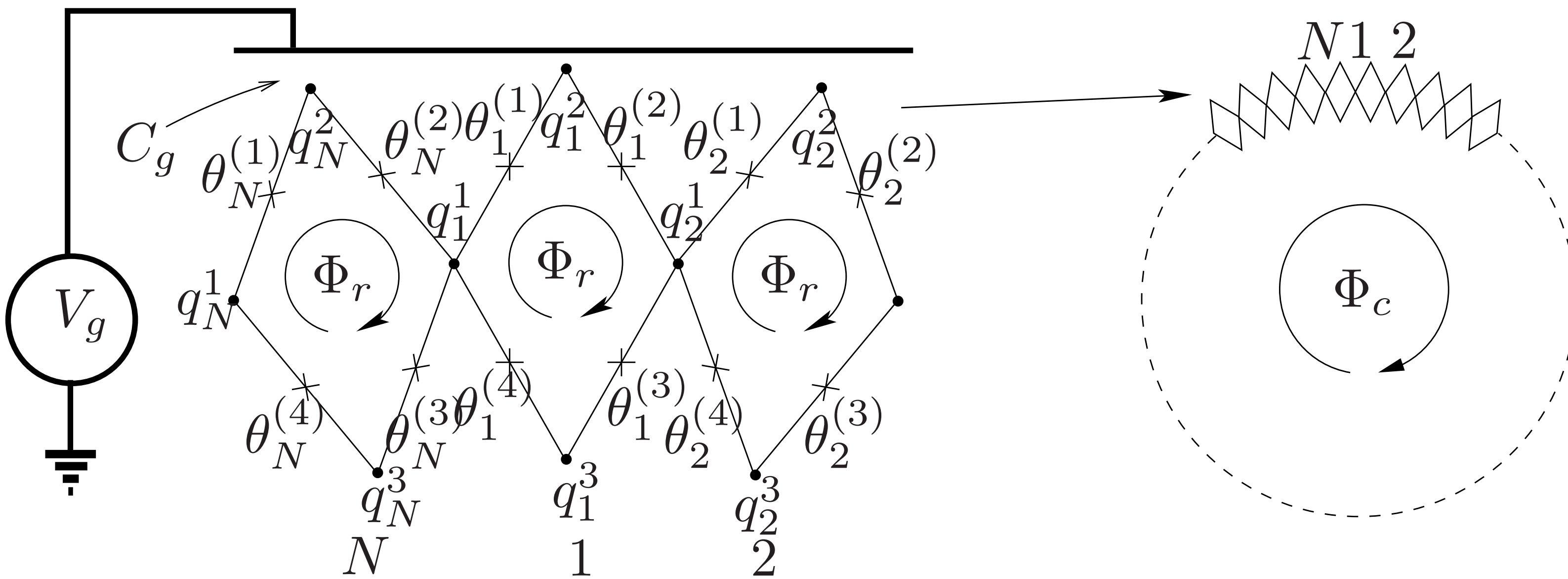


Figure 1: The chain of rhombi with random stray charges. Also shown is an external capacitively coupled gate

$E_J = \hbar I_c^0/2e$ — Josephson energy

$E_C = e^2/2C$ — charging energy, $E_J \gg E_C$

N — number of rhombi

$$2\pi\frac{\Phi_r}{\Phi_0} = \pi + \delta \qquad 2\pi\frac{\Phi_c}{\Phi_0} = \gamma$$

What do we expect???

$\delta = 0 \Longrightarrow$ tunnelling of Cooper pairs along the chain is blocked due to destructive interference of tunneling going through two paths within the same rhombus (B. Douçot & J. Vidal, 2002).

$$I(\gamma) = I_{4e} \sin 2\gamma$$

$\delta \neq 0 \Longrightarrow$ supercurrent has two components

$$I(\gamma) = I_{2e} \sin \gamma + I_{4e} \sin 2\gamma,$$

$I_{2e} = I_{4e} \Longrightarrow |\Phi_r - \Phi_0/2| = \delta\Phi^c$, *crossover point*

In regular chain with $E_J >> E_C$ and no disorder

$$(\delta\Phi^c)_{reg} \approx 0.2 \left(\frac{v}{E_J}\right)^{2/3} \Phi_0 \qquad v \approx 1.2 \left(E_J^3 E_C\right)^{1/4} \exp\left(-1.61\sqrt{\frac{E_J}{E_C}}\right)$$

v — amplitude for quantum phase slip in one contact

• In a classical chain $\delta\Phi^c \sim 1/N$: quantum fluctuations "stabilize" $4e$-supercurrent. In quantum regime both current in the chain are exponentially suppressed by phase fluctuations, but suppression of $2e$-supercurrent is stronger.

• At large E_J/E_C crossover takes place at deviations large as compared to matrix element υ: $\delta\Phi^c/\Phi_0 \gg \upsilon/E_J$

How does disorder affect the pairing effect?

Chain with random stray charges: model and its classical states

$\theta_n^{(m)}$ — phase difference across the m-th junction in the n-th rhombus.

Josephson energy

$$-\sum_{n=1}^{N}\sum_{m=1}^{4} E_J \cos\theta_n^{(m)}$$

Electrostatic energy

$$\sum_{n_1\,,\,n_2=1}^{N}\quad\sum_{k_1\,,\,k_2=1}^{3}\frac{1}{2}\left[C^{-1}\right]_{k_2\,n_2}^{k_1\,n_1}\left(Q_{n_1}^{k_1}-q_{n_1}^{k_1}\right)\left(Q_{n_2}^{k_2}-q_{n_2}^{k_2}\right)$$

Q_n^k — charge of the n-th island in k-th row.
q_n^k — reminiscence of random stray charges.

Assume that charging energy is due to capacitance of *junctions*.
Imaginary-time action

$$S_E = \int dt \sum_{n=1}^{N} \sum_{m=1}^{4} \left\{ \frac{1}{16E_C} \left(\frac{d\theta_n^{(m)}}{dt} \right)^2 - E_J \cos \theta_n^{(m)} \right\} + \delta S$$

$$\delta S = -i \int dt \sum_{n=1}^{N} \sum_{m=1}^{4} p_n^{(m)} \frac{d\theta_n^{(m)}}{dt}$$

$p_n^{(m)}$ — random parameters

δS is just a total derivative. It does not affect classical dynamics. But it gives random phase factors to tunneling amplitudes along different classical trajectories

Additional constrains

$$\sum_{m=1}^{4} \theta_n^{(m)} = \pi + \delta\,; \quad \sum_{n=1}^{N} \left(-\theta_n^{(3)} - \theta_n^{(4)}\right) = \gamma$$

$E_J \gg E_C$

Our strategy:

- Find classical states
- Take into account quantum phase slips

Classification of classical states

$\Phi_r \approx \Phi_0/2 \Longrightarrow$ a *single* rhombus has two classical states with close energies. The states differ in the sign of the supercurrent circulating around the plaquette.

Physically, classical state of chain is characterized by the *global current I* along the chain, and by the *signs of local currents* in rhombi.

Mathematically, classical states are described by binary variables σ_n^z for *each* n-th rhombus and a collective integer variable m.

So we denote classical states $|m, \sigma\rangle$

Classical energy

$$E_{m,\sigma} \approx \frac{E_J\sqrt{2}}{4N}(\widetilde{\gamma} - \pi N/2 - \pi S^z - 2\pi m)^2 - \sqrt{2}\delta S^z E_J + \text{Const.}$$

$$s_n^z = \frac{1}{2}\sigma_n^z\,, \qquad S^z = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^z\,, \qquad \widetilde{\gamma} = \gamma + \frac{N\varphi}{2}\,.$$

This result also shows that in a classical chain $\delta\Phi^c$ scales as $1/N$.

Now we should take into account quantum fluctuations

Quantum fluctuations

$E_C \neq 0 \Longrightarrow$ *quantum phase slips* (QPS) in each Josephson junction $\Longrightarrow$ mixing of different classical states.

υ — amplitude of a QPS in one contact.

$$\upsilon \approx k \left(E_J^3 E_C\right)^{1/4} \exp\left(-1.61\sqrt{E_J/E_C}\right)$$

We should take into account QPS in all junctions. Each QPS brings system from one classical state into another. Each QPS changes "spin" of one rhombi. Due to random stray charges each QPS has its own phase.

$\upsilon \ll E_J \Longrightarrow$ tight-binding approximation

For regular chain

$$\widehat{H}\,|m,\{\sigma_n^z\}\rangle = E_{m,\sigma}\,|m,\{\sigma_n^z\}\rangle + 2\upsilon\sum_{k=1}^{N}\left|m,\{\sigma_1^z,\cdots\sigma_{k-1}^z,-\sigma_k^z,\cdots\sigma_{k+1}^z,\sigma_N^z\}\right\rangle$$

$$+2\upsilon\sum_{k=1\,,\,\sigma_k^z=1}^{N}\left|m+1,\{\sigma_1^z,\cdots\sigma_{k-1}^z,-\sigma_k^z,\cdots\sigma_{k+1}^z,\sigma_N^z\}\right\rangle$$

$$+2\upsilon\sum_{k=1\,,\,\sigma_k^z=-1}^{N}\left|m-1,\{\sigma_1^z,\cdots\sigma_{k-1}^z,-\sigma_k^z,\cdots\sigma_{k+1}^z,\sigma_N^z\}\right\rangle\ .$$

Appropriate tight-binding Hamiltonian looks complicated but after Fourier transformation

$$|x,\sigma\rangle = \sum_m \exp\left\{2i\left(2m-\frac{\widetilde{\gamma}}{\pi}+S^z+\frac{N}{2}\right)x\right\}|m,\sigma\rangle\ ,$$

we obtain Schröedinger equation

$$\frac{\partial^2\psi}{\partial x^2}+\left(\widetilde{E}-2w\cos 2x\sum_{n=1}^{N}a_n\widehat{S}_n^x-2w\sin 2x\sum_{n=1}^{N}b_n\widehat{S}_n^y+2h\widehat{S}^z\right)\psi=0$$

$$\widetilde{E} = \frac{16NE}{\sqrt{2}E_J\pi^2}, \qquad w = \frac{64Nv}{\sqrt{2}E_J\pi^2}, \qquad h = \frac{8N\delta}{\pi^2}.$$

$$\psi = \psi(x, \{\sigma_1 \dots \sigma_n\})$$

a_n and b_n — random coefficients emerging from stray charges

$$a_n = \frac{\cos \pi Q_n^1 + \cos \pi Q_n^2}{2}, \qquad b_n = \frac{\cos \pi Q_n^1 - \cos \pi Q_n^2}{2}$$

$$Q_n^1 = q_n^{(2)} - \frac{1}{3N} \sum_{n=1}^{N} \sum_{k=1}^{3} q_n^{(k)} \qquad Q_n^2 = q_n^{(3)} - \frac{1}{3N} \sum_{n=1}^{N} \sum_{k=1}^{3} q_n^{(k)}$$

Quantum fluctuations $\Longrightarrow$ cos-like part of the potential;

Deviation from $\Phi_r = \Phi_0/2 \Longrightarrow$ field h

$$e^{i\pi \widehat{S}^z + i\pi N/2}\psi\,(x+\pi/2,\sigma) = e^{i\widetilde{\gamma}}\psi(x,\sigma)\,.$$

Generally speaking, in presence of random stray charges even at maximally frustrated point $\Phi_r = \Phi_0/2$ symmetry properties of Schrödinger equation *do not* prohibit $2e$-supercurrent. This is due to the fact that asymmetric realizations of random charges with $q_n^{(2)} \neq q_n^{(3)}$ (and thus $b_n \neq 0$) break symmetry between two tunneling trajectories of a Cooper pair within the same rhombus

How can we deal with our Schrödinger equation containing $2N$ random parameters?

Eliminating spin

Grand partition function

$$Z = \int \mathcal{D}x(\tau)\exp\left(-\int_0^\beta \frac{\dot{x}^2}{2}\right)\prod_{n=1}^{N}\mathrm{Tr}\left[\widehat{U}_n(\beta)\right]$$

Operators $\widehat{U}_n$ act in the state space of spin $\frac{1}{2}$

$$\frac{d\widehat{U}_n}{d\tau} = -\left(wf_n(\tau)S^x + wg_n(\tau)\widehat{S}^y - h\widehat{S}^z\right)\widehat{U}_n$$

$$f_n(\tau) = a_n\cos 2x(\tau)\,, \qquad g_n(\tau) = b_n\sin 2x(\tau)$$

Assume $h \gg w \implies \mathrm{Tr}\,U$ can be calculated explicitly for arbitrary $x(\tau)$

Effective action

$$S = \int_0^\beta d\tau \frac{\dot{x}^2}{2} - A\frac{w^2}{8}\int_0^\beta d\tau_1 d\tau_2 \cos 2x(\tau_1)\cos 2x(\tau_2)\exp(|\tau_1-\tau_2|) -$$

$$B\frac{w^2}{8}\int_0^\beta d\tau_1 d\tau_2 \sin 2x(\tau_1)\sin 2x(\tau_2)\exp(|\tau_1-\tau_2|) -$$

$$iC\frac{w^2}{4h}\int_0^\beta d\tau_1 d\tau_2 \cos 2x(\tau_1)\sin 2x(\tau_2)\exp(|\tau_1-\tau_2|)\,\mathrm{sign}(\tau_1-\tau_2)$$

$$A = \sum_{n=1}^N a_n^2\,, \qquad B = \sum_{n=1}^N b_n^2\,, \qquad C = \sum_{n=1}^N a_n b_n$$

Looks a bit terrific but can be made local via Hubbard-Stratonovich transformation

$$Z = \int_0^\beta \mathcal{D}x(\tau)\mathcal{D}y(\tau)\mathcal{D}z(\tau)e^{-S[x(\tau)]}$$

$$S = h\int d\tau \left(\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} + \frac{y^2 + z^2}{2} + \alpha_1 dy \cos 2x + i\beta_1 d\dot{y} \sin 2x + \right.$$
$$\left. \alpha_2 dz \sin 2x + i\beta_2 d\dot{z} \cos 2x - \frac{d^2}{2}(\beta_1^2 \sin^2 2x + \beta_2^2 \cos^2 2x)\right)$$

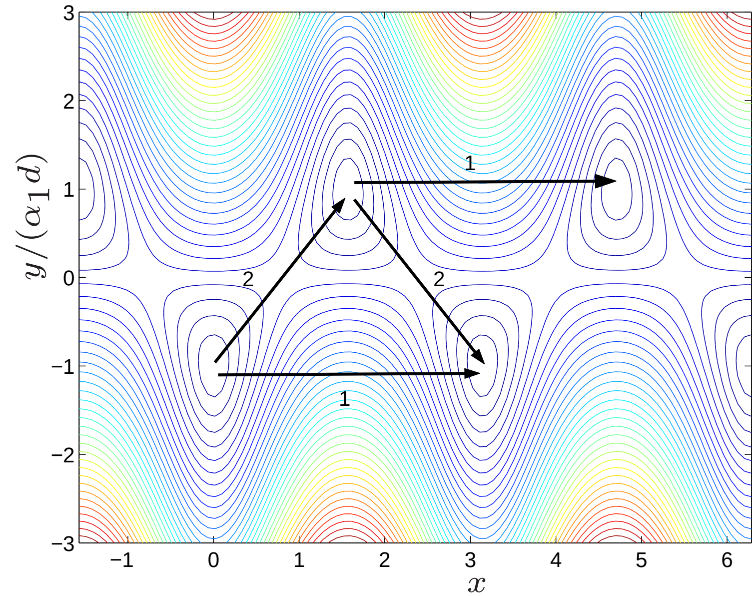
$$\alpha_1^2 + \beta_2^2 = \frac{A}{N}, \quad \beta_1^2 + \alpha_2^2 = \frac{B}{N}, \quad \alpha_1\beta_1 - \alpha_2\beta_2 = \frac{C}{N}, \quad d = \frac{\sqrt{2}\pi}{\delta^{3/2}}\frac{\upsilon}{E_J}$$

We have reduced original problem including large number of random variables ($3N$) to a problem with only *three* random parameters A, B, and C. More over, since A, B and C arise as sums of large number N of independent random variables, it is natural to expect that they obey Gaussian statistics

$$\langle A\rangle = \langle B\rangle = \frac{N}{4} \qquad \left\langle A^2\right\rangle = \left\langle B^2\right\rangle = \frac{N^2}{16} + \frac{5N}{64}$$

$$\langle AB\rangle = \frac{N^2}{16} - \frac{3N}{64} \qquad \langle C\rangle = 0\,, \qquad \left\langle C^2\right\rangle = \frac{N}{64}$$

Amplitudes of supercurrent components



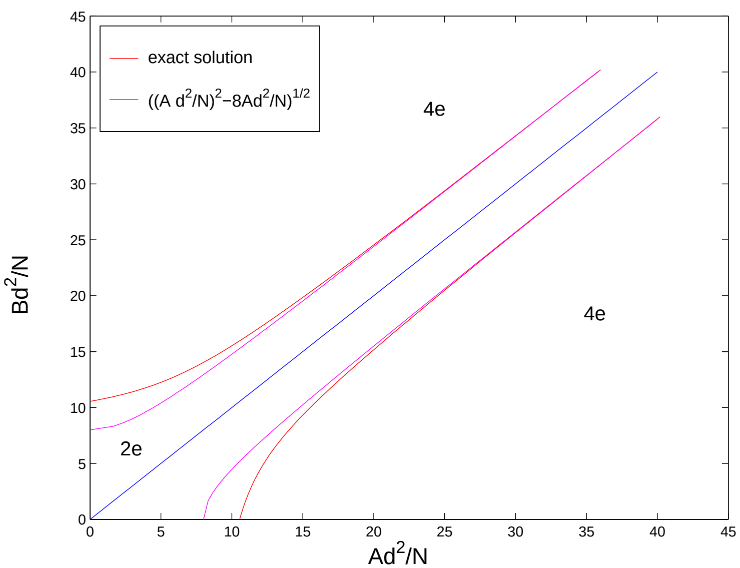
Two types of tunneling trajectories $\Longrightarrow$ two components of supercurrent

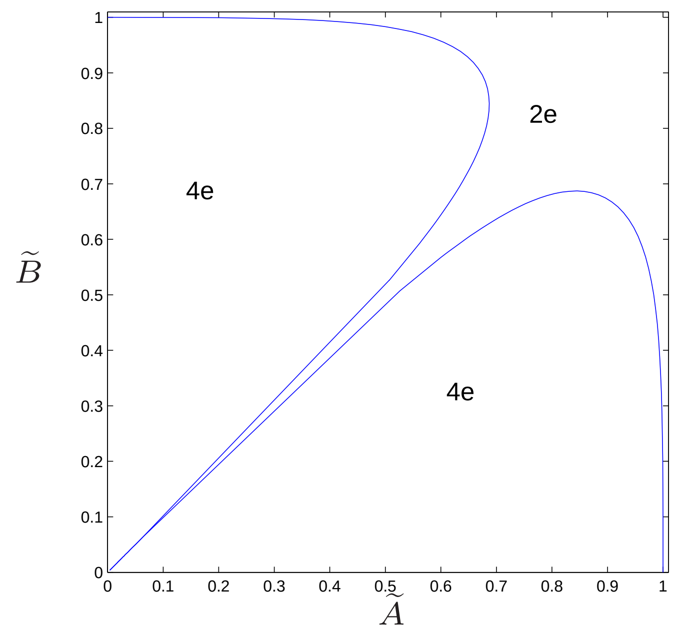
$$I(\gamma) = I_{2e} \sin\gamma + I_{4e} \sin(2\gamma),$$
$$I_{4e} = A_{4e} \exp(-S_E^{4e}), \quad I_{2e} = A_{2e} \exp(-S_E^{2e}),$$

Suppose we know realization of random stray charges. Comparing S_{4e} and S_{2e} we can determine whether we are in $4e$- or $2e$-regim

Crossover line

$$\frac{Bd^2}{N} = \sqrt{\left(\frac{Ad^2}{N}\right)^2 - 8\frac{Ad^2}{N}} \qquad d = \frac{\sqrt{2}\pi}{\delta^{3/2}}\frac{\upsilon}{E_J}$$





$$\widetilde{A} = \frac{\delta\Phi}{(\delta\Phi^c)_{reg}}\left(\frac{NA}{A^2+B^2}\right)^{1/3} \qquad \widetilde{B} = \frac{\delta\Phi}{(\delta\Phi^c)_{reg}}\left(\frac{NB}{A^2+B^2}\right)^{1/3}$$

For any GIVEN set of quenched random charges we can (in principle) determine whether $4e$-supercurrent dominates in the chain. But we need some statistical description of rhombi chain with random charges.

Statistical description of chain and critical deviation

We are interested in probability $\mathcal{P}(E_J/E_C, N, \delta)$ to find dominating $4e$-supercurrent in the chain.

Finally at sufficiently large d we get

$$\mathcal{P}_{4e}(E_J/E_C, N, \delta) = 1 - \mathrm{Erf}\left(\sqrt{\frac{32N}{d^4}}\right)$$

Let us introduce parameter κ and DEFINE critical deviation from the maximally frustrated point $\delta\Phi_c$ as deviation under which the probability above equals κ. Reasonable choice for κ is 0.5 or 0.75 or something else. We then get our final result

$$\frac{\delta\Phi_c}{\Phi_0} = \frac{\left(\mathrm{Erf}^{-1}\left(1-\kappa\right)\right)^{1/6}}{2^{3/2}\pi^{1/3}} \frac{1}{N^{1/6}} \left(\frac{v}{E_J}\right)^{2/3}$$

Compare with clean case

$$(\delta\Phi^c)_{reg} \approx 0.2 \left(\frac{v}{E_J}\right)^{2/3} \Phi_0$$

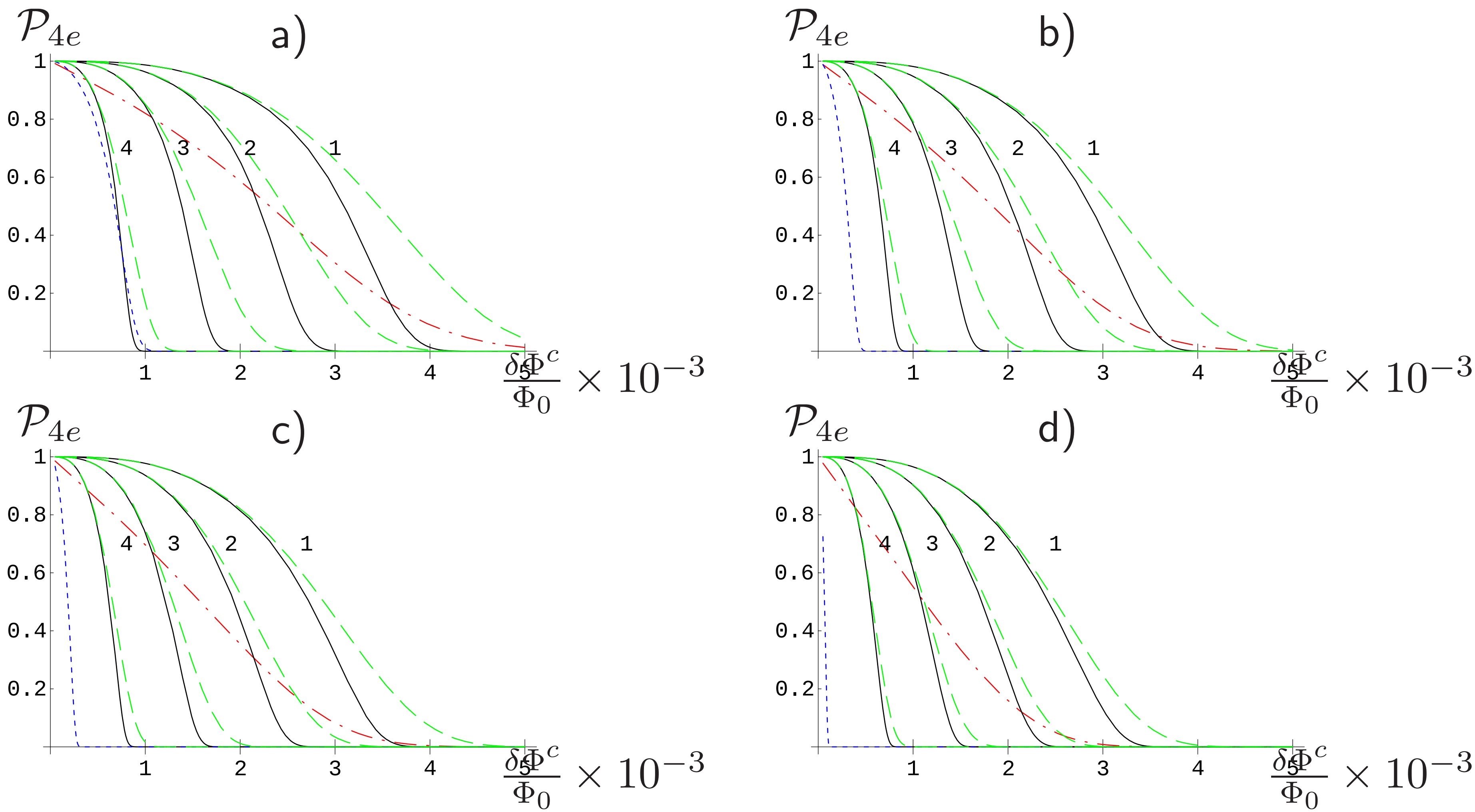


Figure 2: Probability $\mathcal{P}_{4e}$ as function of deviation from maximally frustrated point. Subplots a), b), c), d), correspond to $N = 10, 20, 30, 70$.

Modulation of the supercurrent by capacitive gate

Regular chain:

$$a_n = \frac{1 + \cos \pi C_g V_g}{2} \equiv a\,, \qquad b_n = \frac{1 - \cos \pi C_g V_g}{2} \equiv b$$

Semiclassical description for *total* spin of the chian

Results:

• Change in $4e$-action upon applying external gate voltage is *negative* and proportional to the number of rhombi, i.e. external gate leads to significant increase of otherwise suppressed by fluctuations $4e$-supercurrent.

• $2e$-supercurrent dominates over the current of pairs of Cooper pairs only in small vicinity of the point $C_g V_g / e = 1$.

Disordered chain with gate

Change in $4e$-action upon applying gate voltage is now a random quantity depending on the realization of disorder.

$$\langle \delta S_{4e} \rangle = 0 \qquad \langle (\delta S_{4e})^2 \rangle \sim N$$

• Change in action is now proportional to $\sqrt{N}$ instead of N;

• Variation of the tunnelling action can now be both negative or positive, i.e. applying external voltage we can *decrease* $4e$-supercurrent as well as increase it, depending on the realization of charge disorder.

Any way supercurrent should still be sensitive to external voltage and this can be used to test the quantum nature of the state of the chain

Conclusion

• Rhombi chain with random stray charges was studied. It turns out that problem involving originally large number $(3N)$ of random parameters can be reduced to a much simpler problem with only tree random parameters.

• Statistical description of rhombi chain was obtained

• It is possible to combine low probability of finding significant $2e$-supercurrent at the maximally frustrated point with exponential suppression of the supercurrent by quantum fluctuations

• Main result: pairing effect is rather stable under influence of disorder.

Random fluxes