FREE FIELD APPROACH TO D-BRANES IN GEPNER MODELS.

1. CLOSED STRING COMPACTIFICATION.

Critical Superstring Theory is 10 dimensional. What to do with additional 6 dimensions if we want apply the superstrings for 4 dimensional spacetime?

Answer 1. Kaluca-Klein geometric idea of compactification on some 6-dimensional compact manifold V:

$$W^{10} = R^{1,3} \times V. \qquad (1)$$

The internal string degrees of freedom are given by nonlinear N=1 supersymmetric σ-model on the world-sheet with the action

$$S_{CY} = \int_{\Omega} dz^2 (g_{ij}(Y)\partial Y^i \bar{\partial} Y^j + fermionic\ contribution) \qquad (2)$$

Restrictions on the manifold V:

1) Conformal invariance (string equations of motion):

$$R_{ij} = 0 \tag{3}$$

2) Space-time supersymmetry in 4 dimensions:

V must be complex manifold.

The conditions 1 and 2 mean that 6-dimensional compact manifold must be Calabi-Yau manifold

$$V = CY \tag{4}$$

σ-model has to be N=2 supersymmetric CFT in 2 dimensions.

Answer 2. Algebraic construction of Gepner.

The algebra of symmetries of N=2 supersymmetric CFT is (left and right) N=2 Virasoro superalgebra:

$$[L[n], L[m]] = (n-m)L[n+m] + \frac{n(n^2-1)c}{12}\delta_{n,m}$$
$$[J[n], J[m]] = n\frac{c}{3}\delta_{n,m}$$
$$[L[n], J[m]] = -mJ[n+m]$$
$$\{G^+[r], G^-[s]\} = 2L[r+s] + (r-s)J[r+s] +$$
$$(r^2 - \frac{1}{4})\frac{c}{3}\delta_{r,s}$$
$$[L[n], G^\pm[r]] = (\frac{n}{2} - r)G^\pm[n+r]$$
$$[J[n], G^\pm[r]] = \pm G^\pm[n+r]. \quad (5)$$

$$T_{zz} \equiv T(z) = \sum_n L[n]z^{-n-2},$$
$$J_z \equiv J(z) = \sum_n J[n]z^{-n-1},$$
$$G_z^\pm \equiv G^\pm(z) = \sum_r G^\pm[r]z^{-r-\frac{3}{2}},$$
$$T_{\overline{z}\overline{z}} \equiv \bar{T}(\bar{z}) = \sum_n \bar{L}_[n]\overline{z}^{-n-2}, ... \quad (6)$$

c is central charge of the algebra and one has to demand

$$c = \frac{d}{2} + d = \frac{3}{2}d = 9, \quad (7)$$

where $d = 6$ is the dimension of the compact "manifold".

$$V \approx M_1 \times M_2 \times ... \times M_I, \tag{8}$$

The minimal models are exactly-solvable models of CFT and hence they give exactly-solvable models of superstring compactification.

The Virasoro superalgebra (5) is

$$L[n] = \sum_i L_i[n], \;\; J[n] = \sum_i J_i[n], ... \tag{9}$$

$$c = \sum_{i=1}^{I} c_i = 9$$

$$c_i = 3(1 - \frac{2}{k_i + 2}), \;\; k_i = 0, 1, 2, ... \tag{10}$$

The space of states of each minimal model (NS sector):

$$H_i = \oplus_{(h_i,j_i)}(H_{(h_i,j_i)} \otimes \bar{H}_{(h_i,j_i)}),$$
$$h_i = 0, 1, ..., k_i, \;\; j_i = h_i, h_i - 2, ..., -h_i. \tag{11}$$

$$L_i[n]|h_i,j_i>=J_i[n]|h_i,j_i>=0,\ n>0,$$
$$G_i^{+}[r]|h_i,j_i>=G_i^{-}[r]|h_i,j_i>=0,\ r>0,$$
$$L_i[0]|h_i,j_i>=\frac{h_i(h_i+2)-j_i^2}{4(k_i+2)}|h_i,j_i>,$$
$$J_i[0]|h_i,j_i>=\frac{j_i}{k_i+2}|h_i,j_i>.\qquad(12)$$

The other states from H_{h_i,j_i} can be obtained by the action of Virasoro generators on the highest-weight vectors

$$J_i[-1]|h_i,j_i>\in H_{h_i,j_i}\qquad(13)$$

Thus the space of states of the internal degrees of freedom is

$$H_{int}=\otimes_{i=1}^{I}H_i.\qquad(14)$$

GSO projection:

The space of states (14) is not compatible with our requirement to have supersymmetry in spacetime.

The important fact: spacetime fermions come from R sector representation of N=2 Virasoro superalgebra (r is integer in (5)), spacetime bosons come from NS sector representations (r is half-integer in (5)).

The spectral flow: N=2 Virasoro superalgebra admits the set of automorphisms

$$
\begin{aligned}
G^{+}[r] \rightarrow U^t G^{+}[r] U^{-t} &= G^{+}[r+t], \\
G^{-}[r] \rightarrow U^t G^{-}[r] U^{-t} &= G^{-}[r-t] \\
L[n] \rightarrow U^t L[n] U^{-t} &= L[n] + tJ[n] + \frac{c}{6} t^2 \delta_{n,0}, \\
J[n] \rightarrow U^t J[n] U^{-t} &= J[n] + \frac{c}{3} t \delta_{n,0},
\end{aligned} \tag{15}
$$

where t is a real number and U^t is spectral flow (unitary) operator.

$$
t = \frac{1}{2} : NS \rightarrow R \tag{16}
$$

Then the spacetime supersymmetry operators can be written as

$$
S = S_{st} \prod_i U_i^{\frac{1}{2}}, \;\; \bar{S} = \bar{S}_{st} \prod_i \bar{U}_i^{\frac{1}{2}}, \tag{17}
$$

Spacetime supersymmetric space of states of the string is given by

$$H_{susy} = \sum_{n,m=0}^{2K} S^{\frac{n}{2}} \bar{S}^{\frac{m}{2}} H_{st} \otimes H_{int},$$
$$K = lcm\{k_i + 2\} \qquad (18)$$

$J[0]$-Projection:

$$J[0]|v_{int}>= integer|v_{int}>,$$
$$\bar{J}[0]|v_{int}>= integer|v_{int}>\in H_{int} \qquad (19)$$

In what follows we concentrate on the internal part of the space (18) (in NS sector) because it is obviously most interesting:

$$\hat{H}_{int} = P \sum_{n,m=0}^{K} U^n \bar{U}^m H_{int},$$
$$U = \prod_i U_i, \bar{U} = \prod_i \bar{U}_i, \qquad (20)$$

Question 1: What is the relation between Gepner's algebraic construction and Kaluca-Klein geometric compactification on CY manifolds?

It is clear that some topological properties of CY manifold (the number of zero modes of Dirac operator on CY for example) have to show up in the spectrum of the internal space of states. The corresponding calculations in Gepner models realy reproduce the topological invariants of the CY manifolds!

Question 2: Topological invariants are very rugh characteristic of the manifold. Is there more precise relation between the two approaches?

The explicit answer is not known.

2. OPEN STRING COMPACTIFICATION AND D-BRANES.

2.1. Boundary conditions.

1)N=2 superconformal invariance

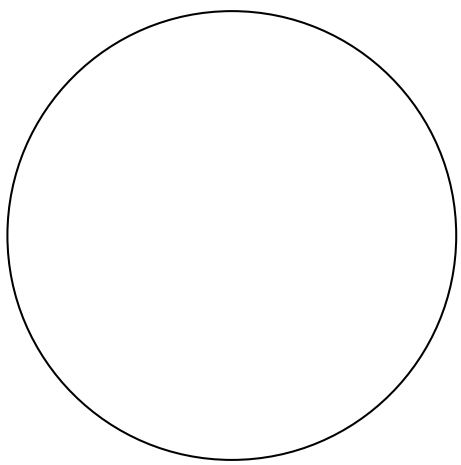
$$T_{zz}(z) = T_{\bar{z}\bar{z}}(\bar{z}),\ J_z(z) = J_{\bar{z}}(\bar{z}),$$
$$G^{+}(z) = \bar{G}^{+}(\bar{z}),\ G^{-}(z) = \bar{G}^{-}(\bar{z}),$$
$$z = \bar{z}. \quad (21)$$

Kaluca-Klein geometric compactification:

These equations impose some relations on the coordinates $Y^i(z, \bar{z})$ which has to be satisfied at the boundary.

Gepner models compactification:

What is geometric interpretation of boundary states in Gepner models? How to treat these equations in Gepner models?



Gluing conditions:

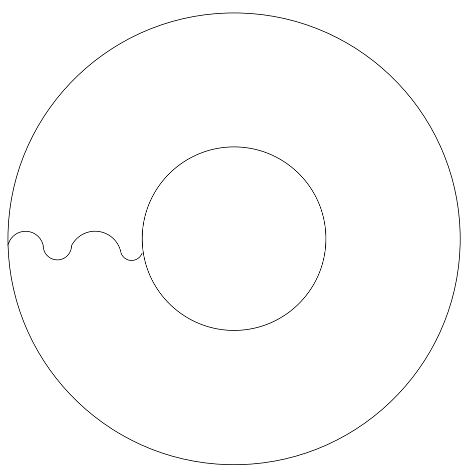
$$\begin{aligned}
(L[n] - \bar{L}[-n])|D>> &= 0, \\
(J[n] + \bar{J}[-n])|D>> &= 0, \\
(G^{+}[r] + \imath\bar{G}^{+}[-r])|D>> &= 0, \\
(G^{-}[r] + \imath\bar{G}^{-}[-r])|D>> &= 0, \\
|D>> &\in H_{int}.
\end{aligned} \qquad (22)$$

2)Spacetime supersymmetry: The overlap with any closed string state which does not belong to $\widehat{H}_{int}$ has to vanish

$$< v|D >>= 0, \ if \ |v >\notin \widehat{H}_{int}. \qquad (23)$$

3)The annulus transition amplitide.

$$< D_1|\exp(-\frac{2\pi T}{l}(L[0] - \frac{c}{24}))|D_2 >= Tr_{H_{12}} \exp(\frac{2\pi l}{T}(L^{op}[0] - \frac{c}{24})) \qquad (24)$$



The amplitude is series of $q = \exp(-\pi \frac{T}{l})$ with integer coefficients.

3. FREEE-FIELD CONSTRUCTION

OF BOUNDARY STATES.

3.1. Free-field realization of individual N=2 minimal model.

$$\partial X(z) = \sum_n X[n] z^{-n-1},$$
$$\partial X^*(z) = \sum_n X^*[n] z^{-n-1},$$
$$\psi(z) = \sum_r \psi[r] z^{-r-\frac{1}{2}},$$
$$\psi^*(z) = \sum_r \psi^*[r] z^{-r-\frac{1}{2}}, \qquad (25)$$

$$[X[n], X^*[m]] = n\delta_{n+m,0},$$
$$\{\psi[r], \psi^*[s]\} = \delta_{r+s,0}, \qquad (26)$$

N=2 Virasoro superalgebra currents

$$G^+(z) = \psi^*(z)\partial X(z) - \frac{1}{\mu}\partial\psi^*(z),$$
$$G^-(z) = \psi(z)\partial X^*(z) - \partial\psi(z),$$
$$J(z) = \psi^*(z)\psi(z) + \frac{1}{\mu}\partial X^*(z) - \partial X(z),$$
$$T(z) = \frac{1}{2}(\partial\psi^*(z)\psi(z) - \psi^*(z)\partial\psi(z)) +$$
$$\partial X(z)\partial X^*(z) - \frac{1}{2}(\partial^2 X(z) + \frac{1}{\mu}\partial^2 X^*(z)), \qquad (27)$$

$$Fock\ module\ F_{p,p^*}$$

$$\begin{aligned}\psi[r]|p,p^* >= \psi^*[r]|p,p^* >= 0, r \geq \frac{1}{2},\\ X[n]|p,p^* >= X^*[n]|p,p^* >= 0, n \geq 1,\\ X[0]|p,p^* >= p|p,p^* >,\\ X^*[0]|p,p^* >= p^*|p,p^* > .\end{aligned} \quad (28)$$

Screening charges:

$$\begin{aligned}Q^+ = \oint dz\psi^* \exp(X^*)(z),\\ Q^- = \oint dz\psi \exp((k+2)X^*)(z),\\ [L[n], Q^{\pm}] = 0, ...\end{aligned} \quad (29)$$

$$(Q^+)^2 = (Q^-)^2 = \{Q^+, Q^-\} = 0. \quad (30)$$

Feigin-Semikhatov Theorem:

$$H_{h,j} = Ker(Q^+ + Q^-)/Im(Q^+ + Q^-) \quad (31)$$

Butterfly resolution for $H_{h,j=h}$:

$$H_{h,j} = U^{\frac{h-j}{2}} H_{h,j=h},$$
$$[Q^{\pm}, U] = 0, \tag{32}$$

U can be represented as an operator acting in Fock modules.

3.2. Free-field realization of the product of minimal models.

$$H_{(\mathbf{h},\mathbf{j})} \approx$$
$$Ker(\sum_i (Q_i^{+} + Q_i^{-}))/Im(\sum_i (Q_i^{+} + Q_i^{-})),$$
$$\mathbf{h} = (h_1, ..., h_I),\ \mathbf{j} = (j_1, ..., j_I). \tag{33}$$

3.3. GSO projection.

Free-field construction of $\widehat{H}_{int}$ can be given directly.

3.4. Free-field construction of boundary states. The solution of gluing equations in free-fields:

$$(\psi_i^*[r] - \imath\bar{\psi}_i^*[-r])|\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^* >>= 0,$$
$$(\psi_i[r] - \imath\bar{\psi}_i[-r])|\mathbf{p}, \mathbf{p}^*, \bar{\mathbf{p}}, \bar{\mathbf{p}}^* >>= 0. \tag{34}$$

$$(X_i[n] + \bar{X}_i[-n] + d_i\delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^* >>= 0,$$
$$(X^*_i[n] + \bar{X}^*_i[-n] + \delta_{n,0})|\mathbf{p},\mathbf{p}^*,\bar{\mathbf{p}},\bar{\mathbf{p}}^* >>= 0, \quad (35)$$

where $d_i = \frac{1}{(k_i+2)}$.

The solution is given by the coherent state

$$|\mathbf{p},\mathbf{p}^* >>= \prod_{n=1} \exp(-\frac{1}{n}\sum_i (X^*_i[-n]\bar{X}_i[-n] + X_i[-n]\bar{X}^*_i[-n])) \prod_{r=1/2} \exp(\imath\eta\sum_i(\psi^*_i[-r]\bar{\psi}_i[-r] + \psi_i[-r]\bar{\psi}^*_i[-r]))|\mathbf{p},\mathbf{p}^*,-\mathbf{p}-\mathbf{d},-\mathbf{p}^*-\mathbf{d}^* >, \quad (36)$$

$\mathbf{d} = (\frac{1}{k_1+2},...,\frac{1}{k_I+2})$, $\mathbf{d}^* = (1,...,1)$.

Boundary states:

The boundary state is given by the linear combination of the coherent states (36) with momentums from the product of butterfly resolutions. The coefficients are fixed by the conditions

1)

$$\sum_i (Q_i^+ + Q_i^-)|D>> = 0, \qquad (37)$$

2)the transition amplitude between a pair of states is open string partition function.

The set of solutions: the boundary states are labeled by the highest-weight states $(\mathbf{h},\mathbf{j}) = (h_1,...,h_I,j_1,...,j_I)$ satisfying GSO projection.

3.5. Open string partition functions and geometry of D-branes.

$$< D_{(\mathbf{h}_1,\mathbf{j}_1)}| \exp(\imath 2\pi\tau(L[0]-\frac{c}{24}))|D_{(\mathbf{h}_2,\mathbf{j}_2)} >= \sum_{(\mathbf{h},\mathbf{j},t)} N_{(\mathbf{h}_1,\mathbf{j}_1)(\mathbf{h}_2,\mathbf{j}_2)(\mathbf{h},\mathbf{j},t)} Tr_{H_{(\mathbf{h},\mathbf{j},t)}} \exp(-\imath\frac{2\pi}{\tau}(L[0]-\frac{c}{24})) \qquad (38)$$

Geometric interpretation.

The Example: $I=5$, $k_i=3$, $K=5$ (Quintic).

The cohomology calculation. First step.

$\sum_i Q_i^+$-cohomology is generated by the set of $bc\beta\gamma$-system of fields

$$a_i(z) = \exp(\mathbf{s}_i^* X(z)),$$
$$a_i^*(z) = (\mathbf{s}_i \partial X^* - \mathbf{s}_i^* \psi \mathbf{s}_i \psi^*) \exp(-\mathbf{s}_i^* X(z)),$$
$$\alpha_i(z) = \mathbf{s}_i^* \psi \exp(\mathbf{s}_i^* X(z)),$$
$$\alpha_i^*(z) = \mathbf{s}_i \psi^* \exp(-\mathbf{s}_i^* X(z)). \quad (39)$$

1)$a_i[0]$ are the coordinates on C^5, $a_i(z)$ are the coordinates on the loop-space of C^5,

2)$a_i^*[0]$ are the derivatives $\frac{\partial}{\partial a_i[0]}$, $a_i^*(z)$ are the derivatives on the loop-space.

3)$\alpha_i[0]$ are the differentails $da_i[0]$ on C^5, $\alpha_i(z)$ are the differentials on the loop-space,....

Thus we can say that the open string propagates in the complex space C^5 with different boundary conditions.

GSO projection:

Z_5 group action on C^5

$$ga_i[0] = \exp(\imath\frac{2\pi}{5})a_i[0] \tag{40}$$

and our space in fact

$$V = C^5/Z_5 \tag{41}$$

Twisted states and orbifold: the string is closed modulo Z_5 transformation.

The second step in cohomology calculation:

$$Q_i^- = \oint dz\alpha_i a_i^{\mu_i - 1}(z), \tag{42}$$

$\sum_i Q_i^-$ is a differential associated with Landau-Ginzburg potential

$$W(a_i) = \sum_i a_i^5. \tag{43}$$

$\sum_i Q_i^-$- cohomology restricts the open string space of states on the surface

$$a_1^5 + ... + a_5^5 = 0. \tag{44}$$