Three-point correlation function in Quantum Toda Field Theory

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Two-dimensional Euclidean field theory

$$\mathcal{L} = \frac{\mathbf{1}}{\mathbf{8}\pi}(\partial\varphi)^{\mathbf{2}} + \mu \sum_{\mathbf{k=1}}^{\mathbf{n-1}} \mathbf{e}^{\mathbf{b}(\mathbf{e_k},\varphi)}$$

e_k are the simple roots of the algebra $sl(n)$ with the matrix of the scalar products $K_{ij} = (e_i, e_j)$

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 & \dots\dots\dots & & 0 \\ -1 & 2 & -1 & \dots\dots\dots & & 0 \\ 0 & -1 & \dots\dots\dots & \dots\dots\dots & \dots & \dots \\ \dots & \dots\dots\dots & \dots\dots\dots & \dots & -1 & 0 \\ 0 & \dots\dots\dots & & -1 & 2 & -1 \\ 0 & \dots\dots\dots & & 0 & -1 & 2 \end{pmatrix}.$$

Instead of two coordinates $\xi = (\xi^1, \xi^2)$ it is convenient to introduce complex coordinates

$$z = \xi^1 + i\xi^2 \quad \bar{z} = \xi^1 - i\xi^2$$

The Lagrangian

There are $n-1$ holomorphic fields $\mathbf{W}^k(z)$

$$\prod_{i=0}^{n-1}(q\partial + (h_{n-i}, \partial\varphi)) = \sum_{k=0}^{n} \mathbf{W}^{n-k}(z)(q\partial)^k$$

where $q = b + 1/b$ and

$$h_k = \omega_1 - e_1 - \cdots - e_{k-1}$$

In particular, $\mathbf{W}^0 = 1$, $\mathbf{W}^1 = 0$ and

$$\mathbf{W}^2 = T(z) = -\frac{1}{2}(\partial\varphi)^2 + (Q, \partial^2\varphi)$$

is the stress-energy tensor of the theory, which ensures local conformal invariance of TFT with central charge

$$c = n - 1 + 12Q^2$$

here $Q = \left(b + \frac{1}{b}\right)\rho$, with ρ being a Weyl vector

$$\rho = \frac{1}{2}\sum_{e>0} e$$

The Lagrangian

The basic objects of the TFT are the exponential fields parametrized by a continuous parameter α

$$V_\alpha = e^{(\alpha,\varphi)}$$

which are the spinless primary fields

$$\mathbf{W}^k(\xi)V_\alpha(z) = \frac{w^{(k)}(\alpha)V_\alpha(z)}{(\xi - z)^k} + \ldots.$$

The quantum numbers $w^{(k)}(\alpha)$ posses Weyl symmetry

$$w^{(k)}(\alpha) = (-1)^k w_s^{(k)}(\alpha) = (-1)^k w^{(k)}(Q + s(\alpha - Q))$$

here s is an element of Weyl group.

In particular,

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2}$$

is the conformal dimension of the field V_α.

Multipoint correlation functions of exponential fields

$$\langle V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)\rangle = \int e^{-\mathcal{A}}\, V_{\alpha_1}(z_1)\dots V_{\alpha_N}(z_N)$$

are the main objects of the theory.

In the particular cases $N=2$ and $N=3$ the correlation functions are determined by the conformal symmetry completely up to the numerical factor

- $N=2$

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle = \begin{cases} \frac{D_s(\alpha_1)}{|z_{12}|^{4\Delta_1}} & \text{if } \alpha_2=Q+s(\alpha_1-Q),\ s\in\mathcal{W} \\ 0 & \text{otherwise} \end{cases}$$

- $N=3$

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{\alpha_3}(z_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2\gamma_{12}}|z_{13}|^{2\gamma_{13}}|z_{23}|^{2\gamma_{23}}}$$

$$\gamma_{12}=\Delta(\alpha_1)+\Delta(\alpha_2)-\Delta(\alpha_3) \quad \text{etc}$$

here $z_{ij}=z_i-z_j$

The constants $\mathrm{D_s}(\alpha)$ and $\mathrm{C}(\alpha_1,\alpha_2,\alpha_3)$ are not determined by the conformal symmetry

sl(3) Example

The algebra of symmetries in this case

$$\mathbf{W}^2(z) = T(z) = \sum \frac{L_n}{z^{n+2}}$$
$$\mathbf{W}^3(z) = W(z) = \sum \frac{W_n}{z^{n+3}}.$$

The Laurent componets L_k and W_k form W^3 algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m},$$

$$[L_n, W_m] = (2n-m)W_{n+m},$$

$$[W_n, W_m] = \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\delta_{n,-m}+$$
$$+\frac{16}{22+5c}(n-m)\Lambda_{n+m} + (n-m)\times$$
$$\times\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m},$$

here

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: + \frac{1}{5}x_n L_n,$$

$$x_{2l} = (1+l)(1-l) \qquad x_{2l+1} = (2+l)(1-l).$$

Operator product expansion (OPE) of V_α with currents

$$T(z)V_\alpha(\xi) = \frac{\Delta(\alpha)V_\alpha(z)}{(z-\xi)^2} + \frac{\partial V_\alpha(z)}{(z-\xi)} + \dots$$

$$W(z)V_\alpha(\xi) = \frac{w(\alpha)V_\alpha(z)}{(z-\xi)^3} + \frac{W_{-1}V_\alpha(z)}{(z-\xi)^2} + \frac{W_{-2}V_\alpha(z)}{(z-\xi)} + \dots$$

here

$$\Delta(\alpha) = \frac{(2Q-\alpha,\alpha)}{2}$$

is the conformal dimension and

$$w(\alpha) = i\sqrt{\frac{48}{22+5c}}\,(\alpha-Q,h_1)(\alpha-Q,h_2)(\alpha-Q,h_3)$$

is the w^3 charge. Here $W_{-1}V(z)$ and $W_{-2}V(z)$ are some new fields. As a consequence of OPE we obtain Ward identities

$$\langle T(z)V_1(z_1)\dots V_n(z_n)\rangle =$$
$$= \sum_{k=1}^{n}\left(\frac{\Delta_k}{(z-z_k)^2} + \frac{\partial_k}{(z-z_k)}\right)\langle V_1(z_1)\dots V_n(z_n)\rangle$$

$$\langle W(z)V_1(z_1)\dots V_n(z_n)\rangle =$$
$$= \sum_{k=1}^{n}\left(\frac{w_k}{(z-z_k)^3} + \frac{W_{-1}^{(k)}}{(z-z_k)^2} + \frac{W_{-2}^{(k)}}{(z-z_k)}\right)\langle V_1(z_1)\dots V_n(z_n)\rangle$$

If the (Δ, w) of the field V_α takes one of the four values

$$\Delta = -\frac{4b^2}{3} - 1 \qquad w^2 = -\frac{2\Delta^2}{27}\frac{5b+\frac{3}{b}}{3b+\frac{5}{b}},$$

$$\Delta = -\frac{4}{3b^2} - 1 \qquad w^2 = -\frac{2\Delta^2}{27}\frac{3b+\frac{5}{b}}{5b+\frac{3}{b}},$$

or in terms of α

$$\alpha = -b\omega_k \qquad \text{or} \qquad \alpha = -\frac{1}{b}\omega_k \qquad\qquad k = 1, 2,$$

the corresponding representation exhibits three null-vectors

$$\chi_1 = \left(W_{-1} - \frac{3w}{2\Delta}L_{-1}\right)V_\alpha = 0,$$

$$\chi_2 = \left(W_{-2} - \frac{12w}{\Delta(5\Delta+1)}L_{-1}^2 + \frac{6w(\Delta+1)}{\Delta(5\Delta+1)}L_{-2}\right)V_\alpha = 0,$$

$$\chi_3 = \left(W_{-3} - \frac{16w}{\Delta(\Delta+1)(5\Delta+1)}L_{-1}^3 + \right.$$
$$\left. + \frac{12w}{\Delta(5\Delta+1)}L_{-1}L_{-2} + \frac{3w}{2\Delta}\frac{(\Delta-3)}{(5\Delta+1)}L_{-3}\right)V_\alpha = 0.$$

Consider for example the correlation function

$$\langle V(z)_{-b\omega_1} V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) V_{\varkappa\omega_2}(z_3)\rangle =$$

$$= |x|^{2b(\alpha_1,h_1)} |1-x|^{\frac{2b\varkappa}{3}} \frac{G(x)}{|z-z_2|^{4\Delta}},$$

with

$$x = \frac{(z_2-z_3)}{(z_1-z_3)} \frac{(z-z_1)}{(z-z_2)} \ .$$

We obtain that $G(x)$ satisfies generalized Pochgamer hypergeometric equation of the type $(3,2)$

$$\left[x\left(x\frac{d}{dx}+A_1\right)\left(x\frac{d}{dx}+A_2\right)\left(x\frac{d}{dx}+A_3\right)-\right.$$
$$\left.-\left(x\frac{d}{dx}+B_1-1\right)\left(x\frac{d}{dx}+B_2-1\right)x\frac{d}{dx}\right]G(x)=0$$

$$x \longrightarrow \bar{x}$$

$$A_k = \frac{b\varkappa}{3} - \frac{2}{3}b^2 + b(\alpha_1 - Q, h_1) + b(\alpha_2 - Q, h_k),$$

$$B_k = 1 + b(\alpha_1 - Q, h_1 - h_{k+1}).$$

$sl(3)$ Example

- $\Psi_k(x),\ k=1,2,3$ basis of solutions of holomorphic equation
- $\bar{\Psi}_k(\bar{x}),\ k=1,2,3$ basis of solutions of antiholomorphic equation

General solution to the both equations $G^{ij}\Psi_i(x)\bar{\Psi}_j(\bar{x})$

Should be single-valued on a sphere with three punctures

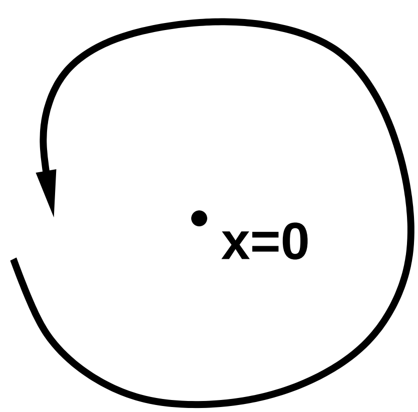
Let $\Psi_k(z)$ is the basis with diagonal monodromy near $x=0$

$$\Psi_k(x) = x^{\lambda_k}\,(1+O(x))$$

The diagonal bilinear combination

$$C_1|\Psi_1(x)|^2 + C_2|\Psi_2(x)|^2 + C_3|\Psi_3(x)|^2 \qquad \star$$

is invariant if we move x around 0



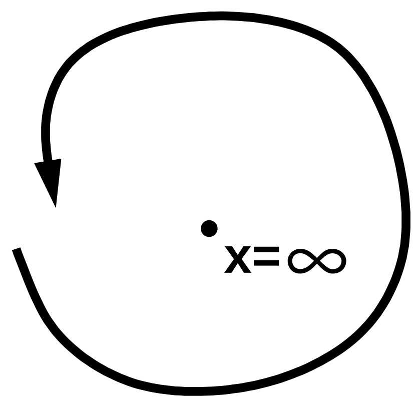
Let $\chi_k(x)$ is the base of solutions to the with diagonal monodromy around ∞

$$\chi_k(x) = \left(\frac{1}{x}\right)^{\tilde{\lambda}_k}\left(1 + O\left(\frac{1}{x}\right)\right)$$

The diagonal bilinear combination

$$\tilde{C}_1|\chi_1(x)|^2 + \tilde{C}_2|\chi_2(x)|^2 + \tilde{C}_3|\chi_3(x)|^2 \qquad \star\star$$

is invariant if we move x around ∞



Both $(\star)$ and $(\star\star)$ should be valid

$$C_1|\Psi_1(x)|^2 + C_2|\Psi_2(x)|^2 + C_3|\Psi_3(x)|^2 =$$
$$= \tilde{C}_1|\chi_1(x)|^2 + \tilde{C}_3|\chi_2(x)|^2 + \tilde{C}_3|\chi_3(x)|^2$$

As $\Psi_k(x)$ and $\chi_k(x)$ satisfy the same equation, they are linearly connected

$$\Psi_i(x) = M_{ij}\chi_j(x)$$

We arrive at the system

$$\begin{pmatrix} M_{11}M_{12} & M_{11}M_{13} & M_{12}M_{13} \\ M_{21}M_{22} & M_{21}M_{23} & M_{22}M_{23} \\ M_{31}M_{32} & M_{31}M_{33} & M_{32}M_{33} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0.$$

or

$$\frac{C_1}{C_2} = \frac{\prod_{k=1}^{3}\gamma(A_k)\gamma(B_1 - A_k)}{\gamma(B_1)\gamma(B_2)}\frac{\gamma(1 - B_1 + B_2)}{\gamma(B_1 - 1)}$$

$$\frac{C_1}{C_3} = \frac{\prod_{k=1}^{3}\gamma(A_k)\gamma(B_2 - A_k)}{\gamma(B_1)\gamma(B_2)}\frac{\gamma(1 - B_2 + B_1)}{\gamma(B_2 - 1)}$$

here

$$\gamma(y) = \frac{\Gamma(y)}{\Gamma(1-y)}$$

$$C_k \sim C(\alpha_1 - bh_k, \alpha_2, \varkappa\omega_2)$$

Hence we obtain functional equation on three-point function $C(\alpha_1, \alpha_2, \varkappa\omega_2)$. The solution

$$C(\alpha_1, \alpha_2, \varkappa\omega_2) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{(2Q-\sum\alpha_i,\rho)}{b}} \times$$
$$\times \frac{(\Upsilon(b))^2\,\Upsilon(\varkappa)\prod_{e>0}\Upsilon((Q-\alpha_1,e))\Upsilon((Q-\alpha_2,e))}{\prod_{ij}\Upsilon\left(\frac{\varkappa}{3}+(\alpha_1-Q,h_i)+(\alpha_2-Q,h_j)\right)},$$

where

$$\log\Upsilon(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{q}{2}-x\right)^2 - \frac{\sinh^2\left(\frac{q}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]$$

$$q = b + 1/b$$