Passive Scalar Evolution in Peripheral Region

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The talk represents results obtained in the works:

M. Chertkov and V. Lebedev,
Phys. Rev. Lett. **90**, 034501 (2003);
M. Chertkov and V. Lebedev,
Phys. Rev. Lett. **90**, 134501 (2003);
V. Lebedev and K. Turitsyn,
Phys. Rev. E**69**, 036301 (2004);
A. Chernykh, V. Lebedev, and K. Turitsyn,
in preparation.

We investigate the passive scalar (concentration of pollutants or temperature) evolution in the random (turbulent) flows near the boundary (wall). If the Schmidt number (which is the relation of the kinematic viscosity to the diffusion coefficient) is large then just the peripheral regions dominate the advanced stages of the passive scalar homogenization (decay). There are some peculiarities of the decay related to the character of the velocity dependence near the wall. However, the decay has **universal features**, and our goal is to establish the features.

Peripheral Regions:

We analyze the passive scalar dynamics in laminar boundary layers characterized by smooth in space and random in time velocity field. Such layers are formed near walls for the cases of developed turbulence or elastic turbulence. The last one is excited in polymer solutions if the characteristic velocity gradient exceeds the inverse polymer relaxation time.

Elastic Turbulence: A. Groisman and V. Steinberg, Nature **405**, 53 (2000); Phys. Rev. Lett. **86**, 934 (2001); Nature **410**, 905 (2001).

Though the velocity is smooth in the peripheral regions it randomly varies in time there.

The passive scalar dynamics (in the absence of pumping) is described by the equation

$$\partial_t \theta + \boldsymbol{v} \nabla \theta = \kappa \nabla^2 \theta \,.$$

In the main approximation the wall can be treated as flat. Let x, z are coordinates along the wall and y is coordinate in the perpendicular direction. The incompressibility condition $\partial_x v_x + \partial_y v_y + \partial_z v_z = 0$, the boundary condition $\boldsymbol{v}|_{y=0} = 0$ and the **smoothness of the velocity** lead to

$$v_y \propto y^2 \,, \quad v_x, v_z \propto y \,.$$

The law is correct for small y/η where η is the width of the boundary layer.

We assume $\mathrm{Sc} = \nu/\kappa \gg 1$. Then mixing in the peripheral region is slow comparing to bulk. The relatively fast mixing in bulk leads to an effective homogenization there, and at the analysis of the peripheral dynamics θ can be treated as constant in bulk. We choose the constant to be equal to zero.

Let us establish **boundary conditions for the passive scalar field**

In the main approximation the wall is flat, $y = 0$.

Concentration of pollutants: $\partial_y \theta|_{y=0} = 0$;

Temperature (fixed on the wall): $\theta|_{y=0} = \vartheta_0$.

$\theta \to 0$ if $y \to \infty$ (that corresponds to bulk where mixing is perfect).

The slow character of the passive scalar evolution in the peripheral region justifies the **turbulent diffusion approximation**. Say, for the first moment of the passive scalar and for its pair correlation function $F(t,\boldsymbol{r}_1,\boldsymbol{r}_2)=\langle\theta(t,\boldsymbol{r}_1)\theta(t,\boldsymbol{r}_2)\rangle$ one obtains

$$\partial_t\langle\theta\rangle=\nabla\left[\hat{D}(\boldsymbol{r},\boldsymbol{r})\nabla\langle\theta\rangle\right]+\kappa\nabla^2\langle\theta\rangle\,,$$

$$\partial_t F(t,\boldsymbol{r}_1,\boldsymbol{r}_2)=\kappa(\nabla_1^2+\nabla_2^2)F$$

$$+\nabla_1\left[\hat{D}(\boldsymbol{r}_1,\boldsymbol{r}_1)\nabla_1 F\right]+\nabla_2\left[\hat{D}(\boldsymbol{r}_2,\boldsymbol{r}_2)\nabla_2 F\right]$$

$$+\nabla_1\left[\hat{D}(\boldsymbol{r}_1,\boldsymbol{r}_2)\nabla_2 F\right]+\nabla_2\left[\hat{D}(\boldsymbol{r}_2,\boldsymbol{r}_1)\nabla_1 F\right]\,,$$

which are closed second-order differential equations.

Here $\hat{D}$ is **turbulent diffusion tensor**. We assume that the flow is statistically homogeneous in time. Then the turbulent diffusion tensor is t-independent and is introduced via the integral

$$D_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_0^\infty \mathrm{d}t \, \langle v_\alpha(t, \boldsymbol{r}_1) v_\beta(0, \boldsymbol{r}_2) \rangle \, .$$

However, it is non-homogeneous in space and strongly anisotropic. Say, $D_{xx} \propto y^2$ whereas $D_{yy} \propto y^4$. Statistical properties of the flow are assumed to be homogeneous along the wall. Then $\hat{D}$ depends on the differences of the longitudinal coordinates like $x_1 - x_2$.

We assume that initially the passive scalar is a large-scale field. Then after the homogenization in space the passive scalar moments are slowly varying along the wall. Say, the first moment $\langle\theta\rangle$ depends on y mainly and we find the equation

$$\partial_t\langle\theta\rangle = \mu\partial_y\left(y^4\partial_y\langle\theta\rangle\right) + \kappa\partial_y^2\langle\theta\rangle\,.$$

The coefficient μ in the equation can be estimated as $\mu\sim\tau V^2/\eta^4$ where V is characteristic velocity fluctuation at $y\sim\eta$ that is on the edge of the boundary layer. Kolmogorov estimates give $\mu\sim\epsilon/\nu^2$, where ϵ is energy flux.

Comparing different terms in the equation for $\langle\theta\rangle$ we find a thickness of the boundary diffusion layer

$$r_{bl} = (\kappa/\mu)^{1/4}\,.$$

For $y < r_{bl}$ the molecular diffusion dominates and for $y > r_{bl}$ the turbulent diffusion dominates. Kolmogorov:

$$r_{bl}/\eta \sim (\kappa/\nu)^{1/4} = \mathrm{Sc}^{-1/4}\,.$$

Thus, $r_{bl} \ll \eta$ at our assumption $\mathrm{Sc} \gg 1$. Note that r_{bl} is much larger than the bulk diffusion length, it is related to slow advection in the direction perpendicular to the wall.

We examine the passive scalar evolution in the peripheral region, which begins after its homogenization in bulk is finished. Then it is natural to expect that the initial distribution of the passive scalar has the characteristic length η in the y direction. The subsequent evolution is divided into two stages. At the first stage the thickness δ of the layer, where θ is concentrated, diminishes as $\delta = (\mu t)^{-1/2}$. When δ reaches r_{bl}, the second stage starts, which is characterized by the fixed spatial scale r_{bl}. Power time dependencies are characteristic of the first stage whereas exponential decay should be observed at the second stage.

First stage (diffusionless). At times when $r_{bl} \ll \delta \ll \eta$ one obtains a universal profile

$$\langle \theta(t,y) \rangle = \vartheta_0 \left[\operatorname{erf}\left(\frac{\delta}{2y}\right) - \frac{\delta}{\sqrt{\pi}\, y} \exp\left(-\frac{\delta^2}{4y^2}\right) \right] .$$

The expression implies contraction of the region occupied by the passive scalar. If $y \gg \delta$ then

$$\langle \theta \rangle \approx \frac{\vartheta_0 \delta^3}{6\sqrt{\pi} y^3} .$$

If $y \ll \delta$ then $\langle \theta \rangle = \vartheta_0$. So, the value of θ is practically unchanged inside the layer $y < \delta$.

Note that though the equation for $\langle\theta\rangle$ is the conservation law, the total amount of the passive scalar in the peripheral region $\int \mathrm{d}y\,\langle\theta\rangle \sim \vartheta_0\delta$ appears to be time dependent. The reason is that the considered solution corresponds to non-zero passive scalar flux directed to large y, i.e. to the bulk, which can be treated as a big reservoir. This flux is $\mu y^4\partial_y\langle\theta\rangle$. Note also that the passive scalar evolution at the first stage is insensitive to the boundary conditions, and therefore it is described identically for the concentration of pollutants and temperature.

Now we analyze the passive scalar behavior at the second stage, then the diffusion term can not be ignored. Let us first examine the case when the passive scalar represents the concentration of pollutants. At long times, only the contribution related to the minimal decay increment is left. That leads to the exponential decay $\langle\theta\rangle \propto \exp(-\gamma t)$. The decrement is $\gamma = 1.81\sqrt{\kappa\mu}$, where the factor is found numerically. The asymptotic behavior of $\langle\theta\rangle$ can be related to the initial value of the passive scalar ϑ_0 near the wall: $\langle\theta\rangle|_{y=0} = 1.55\vartheta_0 \exp(-\gamma t)$. The total amount of the scalar near the boundary behaves as $\int \mathrm{d}y\, \langle\theta\rangle = 1.55\vartheta_0 r_{bl} \exp(-\gamma t)$.

Now we turn to the case where the passive scalar θ represents temperature, assuming that it is fixed at the boundary: $\theta|_{y=0} = \vartheta_0$. Then after the first stage a quasi-stationary distribution of $\langle\theta\rangle$ is formed, since the bulk can be treated as a big reservoir having a constant temperature. This quasi-stationary distribution can be found explicitely

$$\langle\theta\rangle = \frac{2\sqrt{2}}{\pi}\kappa^{3/4}\mu^{1/4}\vartheta_0 \int_y^\infty \frac{\mathrm{d}q}{\mu q^4 + \kappa}.$$

At $y \gg r_{bl}$ we find, again, $\langle\theta\rangle \propto y^{-3}$. That corresponds to a non-zero passive scalar flux (heat flux) to the bulk. This flux is time-independent.

High moments (similar to the first one) depend mainly on y. At the first stage the diffusive term in the equation for the passive scalar correlation functions can be omitted. Then a closed equation for the high moments of the passive scalar can be formulated (analogous to the one for the first moment) which leads to the same universal expression

$$\langle\theta^n\rangle = \vartheta_0^n \left[\mathrm{erf}\left(\frac{\delta}{2y}\right) - \frac{\delta}{\sqrt{\pi}\, y}\exp\left(-\frac{\delta^2}{4y^2}\right)\right],$$

The expression shows that in the region $y \gg \delta$, $\langle\theta^n\rangle \approx \vartheta_0^n \delta^3/(6\sqrt{\pi}\, y^3)$.

At the second stage diffusion starts to be relevant, and it is impossible to obtain closed equations for the moments $\langle\theta^n(t,y)\rangle$. To find the moments one has to solve the complete equations for the passive scalar correlation functions, that is a complicated problem. One can say only that $\langle\theta^n(t,y)\rangle \propto \exp(-\gamma_n t)$ where $\gamma_n \sim \sqrt{\kappa\mu}$. However if $y \gg r_{bl}$ then for high moments we obtain the same equation as for the first one and therefore

$$\langle\theta^n(t,y)\rangle \propto y^{-3}\,.$$

at $y \gg r_{bl}$. The laws correspond to non-zero fluxes of θ^n to bulk.

Now we can turn to the case when the passive scalar is temperature, fixed at the boundary. If the temperature in the bulk is different from that at the boundary then a heat flow is produced from the boundary to the bulk. Since the bulk is a big reservoir then in the main approximation its temperature can be treated as time-independent. In this case statistics of θ becomes quasistationary at the second stage (all the correlation functions are independent of time). Inside the diffusion boundary layer $\theta \sim \vartheta_0$, where ϑ_0 is the temperature at the boundary. For $y \gg r_{bl}$ we have $\langle\theta^n\rangle \sim \vartheta_0^n (r_{bl}/y)^3$.

We established that if $y \gg \delta$ at the first stage or if $y \gg r_{bl}$ at the second stage then the law $\langle \theta^n \rangle \propto y^{-3}$ is valid. It can be treated as a manifestation of an extreme anomalous scaling since the exponents here are independent of n. Moreover, it is possible to formulate the estimates

$$\frac{\langle \theta^n \rangle}{\langle \theta \rangle^n} \sim \left(\frac{y}{\delta}\right)^{3(n-1)}, \quad \frac{\langle \theta^n \rangle}{\langle \theta \rangle^n} \sim \left(\frac{y}{r_{bl}}\right)^{3(n-1)},$$

for the first and second stages. The estimates show that the high moments of the passive scalar are much larger than their Gaussian evaluation, this property implies strong intermittency in the system.

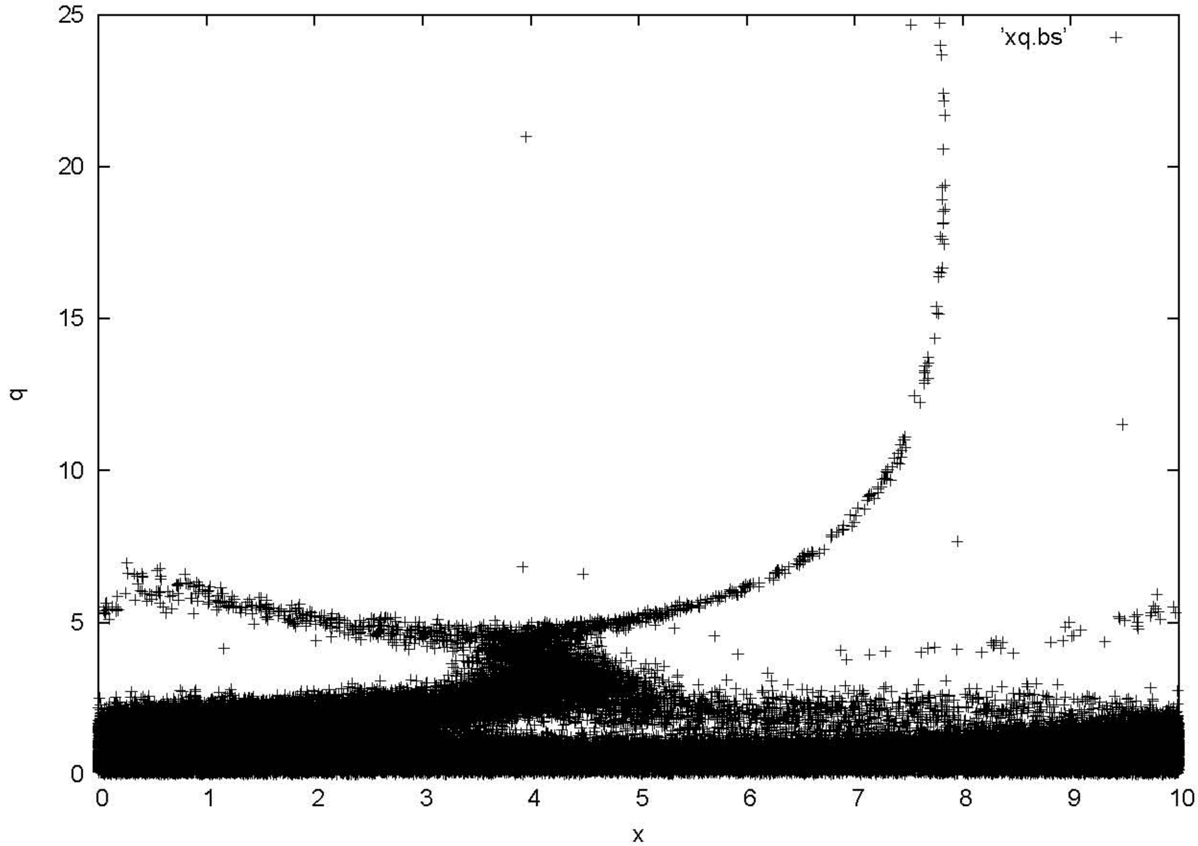

FIG. 1: Typical distribution of pollutant particles near the wall

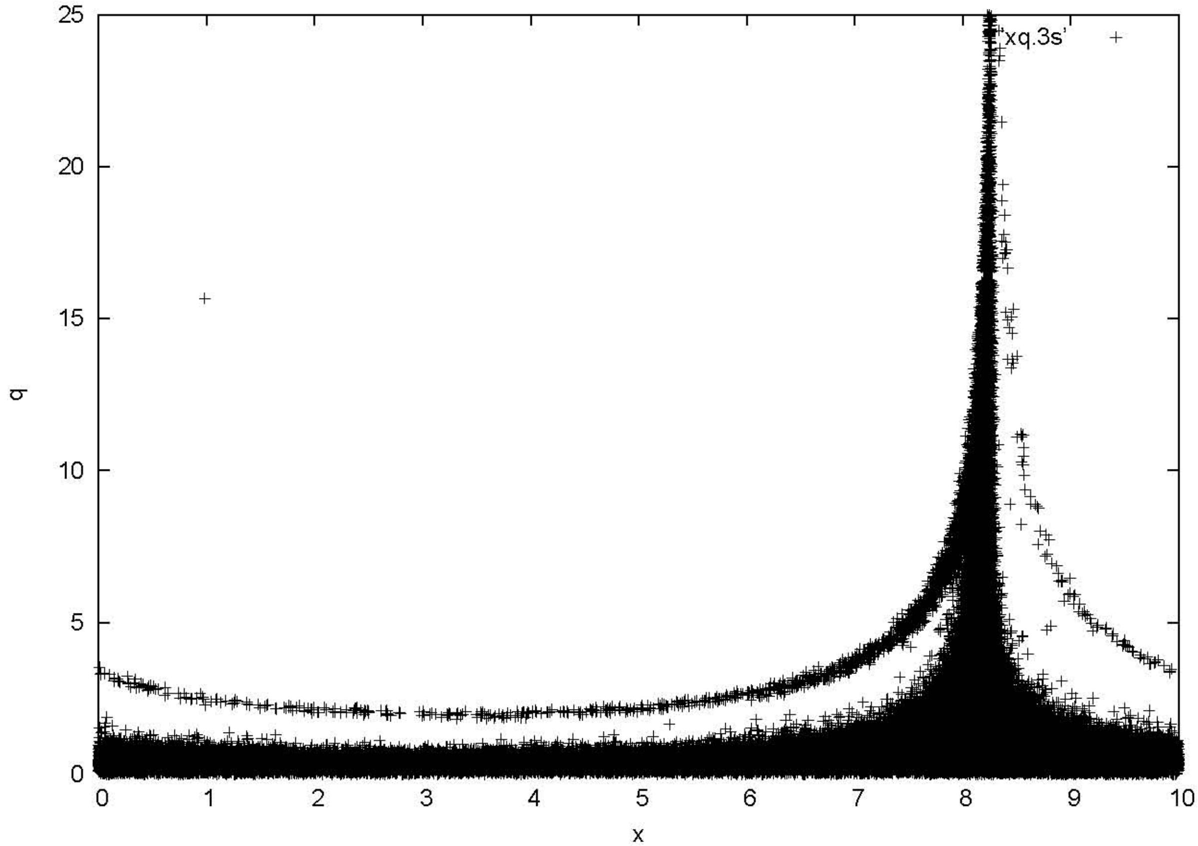

FIG. 2: Typical distribution of pollutant particles near the wall

For $y \gg r_{bl}$ the passive scalar distribution is characterized by long narrow tongues, each tongue is stretched along a Lagrangian trajectory. Fluid particles in a vicinity of the trajectory move in accordance with the equations

$$\partial_t y = \beta_{\alpha\alpha} y^2, \qquad \partial_t \varrho_\alpha = -2y\beta_{\alpha\sigma}\varrho_\sigma,$$

where $\varrho_\alpha = (x, z)$ is the particle separation from the trajectory (along the wall), and $\beta_{\alpha\sigma}$ determines the velocity derivatives. It follows from the equations that for a trajectory bunch its cross-section area behaves as $A \propto y^{-2}$. Therefore the thickness of the tongue diminishes as y increases.

One can introduce the "smoothed" value of the passive scalar

$$\theta_R(\boldsymbol{r}_0) = \frac{3}{4\pi R^3} \int_{|\boldsymbol{r}-\boldsymbol{r}_0|<R} d^3 r \; \theta(\boldsymbol{r})\,.$$

If R is less than the tongue thickness then statistics of θ_R does not differ from that of θ. In the opposite case θ_R has essentially different statistics. Namely,

$$\langle \theta_R^n \rangle \propto R^{2-2n} y^{-1-2n}.$$

We see that the intermittency is "smeared", if we consider the "smoothed" passive scalar. Therefore one should be careful at interpreting experimental or numerical data.

Now we discuss the case when the passive scalar decays along a pipe in a statistically homogeneous flow, which is assumed to be chaotic and has an average velocity u along the pipe. Such setup was used by Groisman and Steinberg in their experiments. In this case the chaotic flow (elastic turbulence) was excited in a polymer solution pushed through a curvilinear pipe. The scalar dynamics is then governed by the equation

$$\partial_t\theta + u\partial_z\theta + \boldsymbol{v}\nabla\theta = \kappa\nabla^2\theta,$$

where $\boldsymbol{v}$ is fluctuating part of the velocity (with zero mean) and z is coordinate along the pipe.

If the pressure difference, pushing the flow, is constant, the flow is statistically stationary and homogeneous along the pipe. Then u is independent of z and the velocity statistics is homogeneous both in t and z. In the turbulent diffusion approximation one obtains for the passive scalar correlation functions

$$u\partial_z F_n = \kappa \sum_{m=1}^{n} \nabla_m^2 F_n + \sum_{m,k=1}^{n} \nabla_m \left[\hat{D} \nabla_k F_n \right],$$

where n is the order of the correlation function. The time derivative is substituted here by the advection term along the pipe.

As previously, we introduce the coordinate y measuring a separation from the wall. The average velocity u behaves $\propto y$ near the wall. Therefore the equation for the first moment turns to

$$sy\partial_z\langle\theta\rangle = \left[\mu\partial_y y^4\partial_y + \kappa\partial_y^2\right]\langle\theta\rangle.$$

Despite the factor y in the left-hand side of the equation, the qualitative picture of evolution remains the same. At the first stage the scalar is mostly situated in the layer of the width $\delta = s/(\mu z)$. The decay at this stage is algebraic with the longitudinal coordinate z. When δ reaches the boundary layer width r_{bl}, the molecular diffusion becomes relevant, and the scalar decay starts to be exponential.

As in the previous case, it is possible to obtain complete statistical properties of the scalar at the first stage when the molecular diffusion is negligible. Then the equations for the passive scalar moments are identical. Solving the equations we find a universal profile

$$\langle \theta^n(z,y) \rangle = \frac{\vartheta_0^n \delta^3}{6y^3} \exp\left(-\delta/y\right)\ {}_1F_1\left(1,4,\delta/y\right)\ .$$

If $y \gg \delta$ then both the exponent and ${}_1F_1$ can be substituted by unity to obtain $\langle \theta^n(z,y) \rangle \approx \vartheta_0^n \delta^3/(6y^3)$. If $y \ll \delta$ then $\langle \theta^n(z,y) \rangle \approx \vartheta_0^n$.

When δ diminishes down to r_{bl} another regime comes. For the density of pollutants the regime is characterized by an exponential decay of the passive scalar moments along the pipe:

$$\langle\theta^n\rangle \propto \exp(-\alpha_n z)\,,$$

where $\alpha_n \sim \kappa^{1/4}\mu^{3/4}s^{-1}$. For the first moment the eigenvalue problem can be solved numerically, then one obtains $\alpha \approx 3.72\kappa^{1/4}\mu^{3/4}s^{-1}$. The law $\alpha_n \propto \kappa^{1/4}$ was checked experimentally for the passive scalar decay in the elastic turbulence case:

T. Burghelea, E. Serge, and V. Steinberg, Phys. Rev. Lett. **92**, 164501 (2004).

All properties of the "smoothed" passive scalar moments established for the decay in time are valid also for the decay in space. The reason is that the tongue (filament) formation is relatively fast process which is realized irrespective to the character of the passive scalar decay. At $y \gg r_{bl}$ the tongues are well separated that leads to the behavior

$$\langle \theta_R^n \rangle \propto R^{2-2n} y^{-1-2n}.$$

already discussed for the passive scalar decay in time.

At our analysis, we neglected an inhomogeneity of the passive scalar along the wall. The inhomogeneity can be involved into consideration. It does not change any qualitative feature. Particularly, the picture of tongues and consequences of the picture remain the same. However, the wall inhomogeneity could influence essentially the character of the passive scalar decay. Say, angles or craters lead to some peculiarities of the passive scalar evolution near the defects. It is a subject of separate investigation.

The peripheral region supplies the passive scalar to the bulk. Because of the slow dynamics, it can be considered as a quasi-stationary stochastic source of the passive scalar for the bulk, where the passive scalar correlation functions adjust adiabatically to the level of the supply. This prediction seems to be in agreement with experimental data

T. Burghelea, E. Serge, and V. Steinberg,

Phys. Rev. Lett. **92**, 164501 (2004),

where a space dependence of the correlation functions close to logarithmic one was observed. It is just the Batchelor-Kraichnan behavior in the smooth velocity field (characteristic of the elastic turbulence).

The final remark concerns an extension of our results to other problems. As it was noted in the paper

M. Chertkov and V. Lebedev,

Phys. Rev. Lett. **90**, 134501 (2003);

the scheme developed for the passive scalar decay can be without serious modifications applied to fast binary chemical reactions. We believe that minor modifications of the scheme can make it applicable for more complicated chemical reactions. The other problem, which can be posed for the peripheral region, is the stochastic polymer dynamics (say, for the case of the elastic turbulence).