Scaling limits is SOS models and form factors

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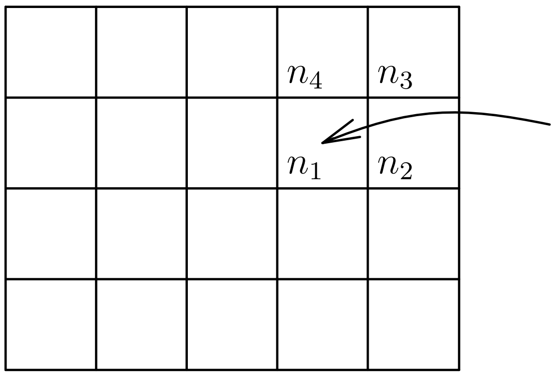
The SOS models



$$W\begin{bmatrix} n_4 & n_3 \\ n_1 & n_2 \end{bmatrix}$$

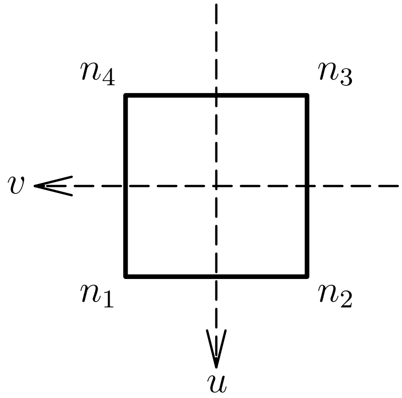
$$n_i \in \mathbb{Z} + \delta$$

Admissibility condition:

$$|n_1 - n_2| = 1,$$
$$|n_2 - n_3| = 1,$$
$$|n_3 - n_4| = 1,$$
$$|n_4 - n_1| = 1.$$

$$Z = \sum_{C:\ \text{configurations}} \prod_{p:\ \text{plaquets}} W\begin{bmatrix} n_4(C,p) & n_3(C,p) \\ n_1(C,p) & n_2(C,p) \end{bmatrix}$$

Integrable family of weights



$$= W \left[\begin{matrix} n_4 & n_3 \\ n_1 & n_2 \end{matrix} \middle| u - v \right]$$

$$W \left[\begin{matrix} n & n \pm 1 \\ n \pm 1 & n \pm 2 \end{matrix} \middle| u \right] = R_0(u),$$

$$W \left[\begin{matrix} n & n \pm 1 \\ n \pm 1 & n \end{matrix} \middle| u \right] = R_0(u) \frac{[n \pm u][1]}{[n][1-u]},$$

$$W \left[\begin{matrix} n & n \pm 1 \\ n \mp 1 & n \end{matrix} \middle| u \right] = R_0(u) \frac{[n \pm 1][u]}{[n][1-u]}$$

$$[u] \sim \theta_1(u/r; \mathrm{i}\pi/\epsilon r), \qquad \theta_1(u;\tau) = \frac{1}{2\mathrm{i}} \sum_{n \in \mathbb{Z}+1/2} (-1)^{n-1/2} \mathrm{e}^{\mathrm{i}\pi\tau n^2 + 2\pi \mathrm{i} n u}$$

Parameters

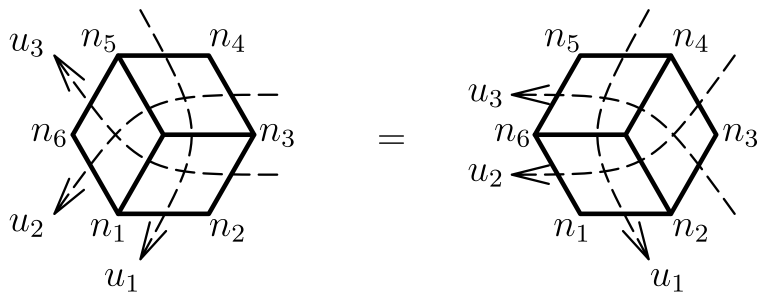
$\boxed{\epsilon}$ is the temperature parameter

$$\epsilon \to \infty \Leftrightarrow T \to 0, \qquad \epsilon \to 0 \Leftrightarrow T \to T_c$$

$\boxed{r}$ is the parameter governing the critical exponents. The limit $r \to \infty$ corresponds to the six-vertex (ice) model.

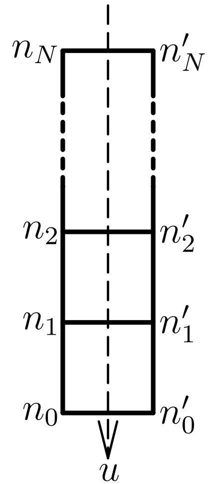
$\boxed{u}$ is the *spectral parameter*.

The Yang–Baxter equation



Transfer matrix

$$T(u)^{n'_0 n'_1 n'_2 \ldots n'_N}_{n_0 n_1 n_2 \ldots n_N} =$$



YB equation $\Rightarrow$ **Commutativity of transfer matrices** for the cyclic boundary conditions ($n_N = n_0$, $n'_N = n'_0$):

$$\boxed{[T(u), T(v)] = 0 \qquad \forall u, v.}$$

$\Rightarrow$ Integrals of motion necessary for integrability.

Restrictions compatible with the YB equation:

1. For $\delta = 0$ and an arbitrary r:

$$\mathrm{RSOS}^{(1)}: \; n = 1, 2, 3, \ldots$$

2. For $\delta = 0$ and rational $r = q/(q-p)$ $(p, q > 0$ are coprimes):

$$\mathrm{RSOS}^{(2)}: \; n = 1, 2, \ldots, q-1.$$

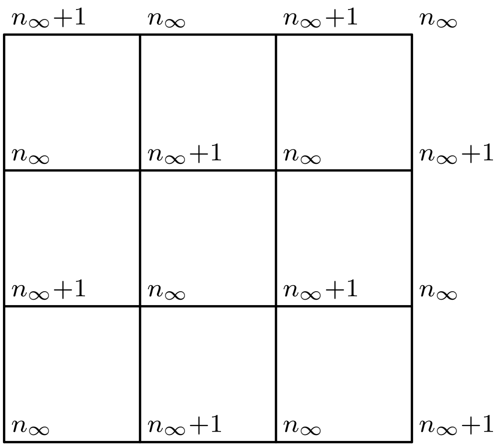
Regime III:

$$\epsilon > 0, \qquad r \geq 1, \qquad 0 < u < 1.$$

For $r = 4, 5, 6, \ldots$ the the Boltzmann weights of the $\mathrm{RSOS}^{(2)}$ are **positive**.

Example: $r = 4$: the Ising model.

Ground states $(T \to 0 \text{ or } \epsilon \to \infty)$



For

$$Nr < \operatorname{Re} n_\infty < (N+1)r - 1$$

define

$$m = n_\infty - N.$$

Space of states: $\mathcal{H}^{(0)}_{(m,m')}$ is spanned by the vectors ('paths') $|\{n_k\}_{k=-\infty}^{\infty}\rangle$ such that

1. $|n_{k+1} - n_k| = 1 (\forall k)$ (admissibility).

2. $n_{2k} = n_\infty, n_{2k+1} = n_\infty + 1$ or $n_{2k} = n_\infty + 1, n_{2k+1} = n_\infty$ for $-k \gg 1$; the similar with n'_∞ for $k \gg 1$ (condition at infinity).

Restricted spaces of states:

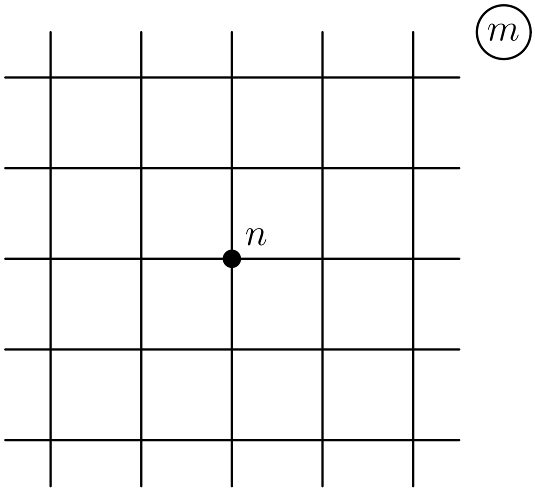
1. $\delta = 0$, $\boxed{m > 0}$:

$$\mathcal{H}^{(1)}_{(m,m')} \subset \mathcal{H}^{(0)}_{(m,m')}, \text{ spanned on paths with } \boxed{n_k > 0}.$$

2. $\delta = 0$, $r = q/(q-p)$, $\boxed{0 < m < p}$:

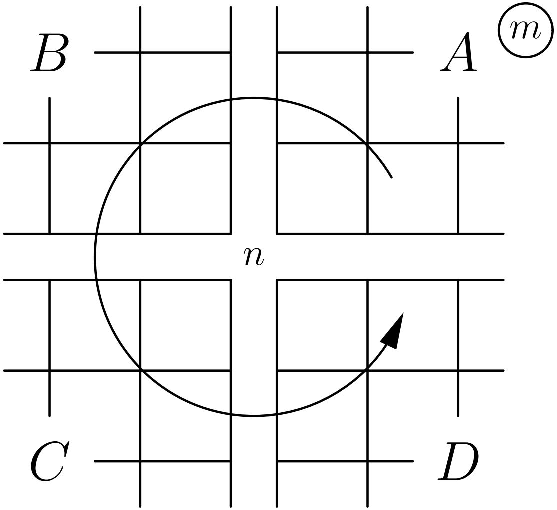
$$\mathcal{H}^{(2)}_{(m,m')} \subset \mathcal{H}^{(0)}_{(m,m')}, \text{ spanned on paths with } \boxed{0 < n_k < q}.$$

Local height probabilities

$Z_{mn} =$ 

$$P_{mn} = \frac{Z_{mn}}{Z_m} = \frac{Z_{mn}}{\sum_n Z_{mn}}$$

Corner transfer matrices



$$Z_{mn} \sim \mathrm{Tr}_{\mathcal{H}_{mn}}(DCBA)$$

Half line spaces: $\mathcal{H}^{(0)}_{mn}$ is spanned on the paths $|\{n_k\}_{k=0}^{\infty}\rangle$ satisfying the conditions:

1. Admissibility $|n_{k+1} - n_k| = 1$.
2. Condition at infinity (stabilization to $\ldots, n_\infty, n_\infty + 1, n_\infty, n_\infty + 1, \ldots$).
3. $n_0 = n$.

The spaces $\mathcal{H}^{(1)}_{mn}$ and $\mathcal{H}^{(2)}_{mn}$ satisfy also the reduction conditions.

The corner Hamiltonian H:

$$DCBA \sim [n]\mathrm{e}^{-4H\epsilon}, \qquad \mathrm{Spec}\, H = \{\Delta_{mn}, \Delta_{mn}+1, \Delta_{mn}+2, \ldots\}.$$

More precizely:

$$\begin{aligned}
\chi^{(0)}_{mn} &\equiv \mathrm{Tr}_{\mathcal{H}^{(0)}}\, z^H = \frac{z^{\Delta_{mn}}}{\prod_{k=0}^{\infty}(1-z^k)}, \\
\chi^{(1)}_{mn}(z) &\equiv \mathrm{Tr}_{\mathcal{H}^{(1)}}\, z^H = \chi^{(0)}_{mn}(z) - \chi^{(0)}_{m,-n}(z), \\
\chi^{(2)}_{mn}(z) &\equiv \mathrm{Tr}_{\mathcal{H}^{(2)}}\, z^H = \sum_{k\in\mathbb{Z}} (\chi^{(0)}_{m,n+2qk}(z) - \chi^{(0)}_{m,-n+2qk}(z)),
\end{aligned}$$

with

$$\Delta_{mn} = \frac{(rm-(r-1)n)^2-1}{4r(r-1)}.$$

For $m, n \in \mathbb{Z}$ they are the **conformal dimensions** in the conformal model with the central charge of the Virasoro algebra

$$c = 1 - \frac{6}{r(r-1)}.$$

The functions $\chi^{(1)}_{mn}(z)$ and $\chi^{(2)}_{mn}(z)$ are the **characters of irreducible representations of the Virasoro algebra**.

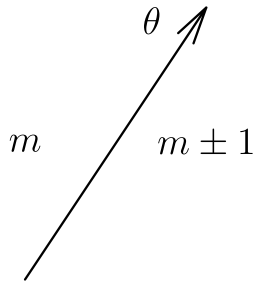
The local height probabilities are given by

$$h_{mn}^{(i)} = [n]\frac{\chi_{mn}^{(i)}}{\chi_m}, \qquad \chi_{mn}^{(i)} \equiv \chi_{mn}^{(i)}(\mathrm{e}^{-4\epsilon}),$$

$$\chi_m = \sum_n [n]\chi_{mn}^{(i)} = [m]'\chi, \qquad \chi = \frac{2\mathrm{e}^{\epsilon/r(r-1)}}{\prod_{k=0}^{\infty}(1-\mathrm{e}^{-2\epsilon(2k+1)})}$$

with $[u]' = [u]|_{r\to r-1}$.

‘Elementary’ excitations: kinks change the vacuum



$$T(u)|\theta_1,\ldots,\theta_N\rangle_{m_0 m_1 \ldots m_{N-1} m_N} = \prod_{i=1}^{N} \tau(\theta_i + \mathrm{i}\pi u)|\theta_1,\ldots,\theta_N\rangle_{m_0 m_1 \ldots m_{N-1} m_N}$$

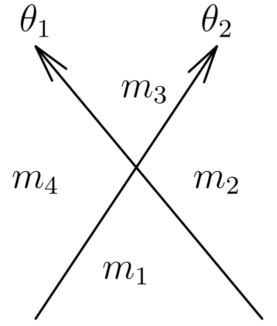
$$\tau(\theta) = \frac{\theta_4\left(\frac{1}{4} + \frac{\theta}{2\pi\mathrm{i}}; \frac{\mathrm{i}\pi}{2\epsilon}\right)}{\theta_4\left(\frac{1}{4} - \frac{\theta}{2\pi\mathrm{i}}; \frac{\mathrm{i}\pi}{2\epsilon}\right)}.$$

In the visinity of the critical point $\epsilon \to 0$:

$$\tau(\theta) = 1 + \mathrm{i}M \,\mathrm{sh}\,\theta + O(M^2), \qquad M = 4\mathrm{e}^{-\pi^2/2\epsilon}.$$

Scattering matrix

$$S\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle| \theta_1 - \theta_2 \right] =$$



$$S\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle| \theta \right] = -W\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle| \frac{\theta}{\mathrm{i}\pi} \right]\Bigg|_{r \to r-1}$$

It satisfies the YB equation.

Vertex operators $\Psi^*(\theta)_m^{m'} : \mathcal{H}^{(i)}_{mn} \to \mathcal{H}^{(i)}_{m'n}$,

$$|\theta_1, \dots, \theta_N\rangle_{m_0 m_1 \dots m_{N-1} m_N} \in \mathcal{H}^{(i)}_{(m_0, m_N)} \quad \leftrightarrow \quad \Psi^*(\theta_N)^{m_N}_{m_{N-1}} \dots \Psi^*(\theta_1)^{m_1}_{m_0}$$

They satisfy the algebra

$$[\tilde{H}, \Psi^*(\theta)^{m'}_m] = \mathrm{i}\frac{d}{d\theta}\Psi^*(\theta)^{m'}_m, \qquad \tilde{H} = 2\epsilon H/\pi,$$

$$\Psi^*(\theta_1)^{m'}_s \Psi^*(\theta_2)^s_m = \sum_{s'} S\left[\begin{matrix} m' & s' \\ s & m \end{matrix}\middle| \theta_1 - \theta_2\right] \Psi^*(\theta_2)^{m'}_{s'} \Psi^*(\theta_1)^{s'}_m,$$

$$\Psi^*(\theta')^{m'}_{m''}\Psi^*(\theta)^{m''}_m = -\frac{\mathrm{i}[m'']'}{\theta' - \theta - \mathrm{i}\pi}\delta_{m'm} + O(1) \text{ as } \theta' \to \theta + \mathrm{i}\pi.$$

Form factors

$$_{mm'_1 \dots m'_{N'-1} m'}\langle\theta'_1 \dots \theta'_{N'}|\Pi_n|\theta_1 \dots \theta_N\rangle_{mm_1 \dots m_{N-1} m'} =$$

$$= \frac{1}{[m]'\chi} \mathrm{Tr}_{\mathcal{H}^{(i)}_{mn}} \Big(\mathrm{e}^{-2\pi\tilde{H}} \Psi^*(\theta'_1 + \mathrm{i}\pi)^m_{m'_1} \dots \Psi^*(\theta'_{N'} + \mathrm{i}\pi)^{m'_{N'-1}}_{m'} \times$$

$$\times \Psi^*(\theta_N)^{m'}_{m_{N-1}} \dots \Psi^*(\theta_1)^{m_1}_m \Big)$$

Bosonization scheme

$$\Psi^*(\theta)_m^{m+1} = :\mathrm{e}^{\mathrm{i}\varphi(\theta)}:,$$

$$\Psi^*(\theta)_m^{m-1} = \eta^{-1}:\mathrm{e}^{\mathrm{i}\varphi(\theta)}: \int_C \frac{d\gamma}{2\pi} :\mathrm{e}^{-\mathrm{i}\varphi(\theta+\mathrm{i}\pi)-\mathrm{i}\varphi(\theta-\mathrm{i}\pi)}: F_m(\gamma-\theta)$$

with

$$F_m(\gamma) = \frac{[\gamma/\mathrm{i}\pi + 1/2 - m]'}{[\gamma/\mathrm{i}\pi - 1/2]'},$$

$$\varphi(\theta) = \sqrt{\frac{r}{2(r-1)}}(\mathcal{Q} + 2\epsilon\mathcal{P}\theta/\pi) + \sum_{k\neq 0} \frac{a_k}{\mathrm{i}k}\mathrm{e}^{2\mathrm{i}\epsilon\theta k/\pi},$$

$$[\mathcal{P},\mathcal{Q}] = -\mathrm{i}, \qquad [a_k, a_{k'}] = \delta_{k+k',0}kA_k, \qquad A_k = \frac{\operatorname{sh}\epsilon k \operatorname{sh}\epsilon r k}{\operatorname{sh} 2\epsilon k \operatorname{sh}\epsilon(r-1)k}.$$

The space $\mathcal{H}_{mn}^{(0)}$ is generated by a_{-k} $(k>0)$ from the vacuum $|P_{mn}\rangle$:

$$a_k|P_{mn}\rangle = 0, \qquad \mathcal{P}|P_{mn}\rangle = P_{mn}|P_{mn}\rangle,$$

$$P_{mn} = m\sqrt{\frac{r}{2(r-1)}} - n\sqrt{\frac{r-1}{2r}}.$$

The corner Hamiltonian

$$H = \frac{\mathcal{P}^2}{2} - \frac{1}{4r(r-1)} + \sum_{k=1}^{\infty} A_k^{-1} a_{-k} a_k.$$

Note that

$$\frac{P_{mn}^2}{2} - \frac{1}{4r(r-1)} = \Delta_{mn}$$

and each a_{-k} increases the ‘energy’ by k.

Scaling limits

$$\begin{aligned} \mathrm{I}: \ \epsilon \longrightarrow 0, \qquad & \delta = \text{const}\,, \\ \mathrm{II}_{\pm}: \ \epsilon \longrightarrow 0, \qquad & \delta = \pm\frac{\mathrm{i}\pi}{2r\epsilon}. \end{aligned}$$

Limit I:

$$[u]' \sim \sin\frac{\pi u}{r-1}$$

and $S\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle|\theta\right]$ tends to the S matrix of the $\phi_{1,3}$ **perturbation of the conformal model** with $c = 1 - 6/r(r-1)$.

Limits $\mathbf{II}_{\pm}$:

$$[u]' \sim \sin\frac{\pi u}{r-1}, \qquad [m'] \sim \mathrm{e}^{\mp\pi m/(r-1)}$$

and $S\left[\begin{matrix} m_4 & m_3 \\ m_1 & m_2 \end{matrix}\middle|\theta\right]$ tends (up to a simple removable factor) to the S matrix of the **sine-Gordon model** with the action

$$\mathcal{A} = \int d^2x \left(\frac{(\partial_\mu\varphi)^2}{8\pi} + \mu\cos\beta\varphi\right), \qquad \beta = \sqrt{2\frac{r-1}{r}}.$$

What are the local height operators Π_{mn} in this limit? Their vacuum expectations

$$P_{mn} \sim \sin\frac{\pi n}{r}.$$

This is not interesting. But consider the Fourier transforms

$$\Pi_m(a) = \sum_n \frac{e^{i\pi an/r}}{\sin\frac{\pi n}{r}}\Pi_{mn}, \qquad -r < \operatorname{Re} a + \frac{2\epsilon}{\pi}\operatorname{Im} m < r.$$

In the **limit I** their vacuum expectation values are

$$P_m(a) \simeq \frac{e^{i\pi\frac{am}{r-1}}}{\sin\frac{\pi m}{r-1}}\left(\frac{M}{4}\right)^{\frac{a^2-1}{2r(r-1)}}.$$

It means that the conformal dimension of this operator is

$$\Delta(a) = \frac{a^2-1}{4r(r-1)}.$$

In the **limit II$_-$**:

$$P_m(a) \sim M^{(a-1)^2/2r(r-1)}$$

It allows to identify these operators (up to a constant) to the exponentials $\mathrm{e}^{\mathrm{i}\alpha\varphi(x)}$ with

$$\alpha = \frac{1-a}{\sqrt{2r(r-1)}}.$$

In the **limit I** we expect to identify the operators $\Pi_m(a)$ with some conformal operators. But first we must make the restrictions.

It is not easy to caculate the restricted sums over n for the traces like $\mathrm{Tr}_{\mathcal{H}^{(0)}_{mn}}(\Phi_{mn}X)$.

Consider the operator

$$\tilde{\Pi}_m(a) = \frac{\Pi_m(a) - \Pi_m(-a)}{2\mathrm{i}} = \sum_n \frac{\sin\frac{\pi a n}{r}}{\sin\frac{\pi n}{r}} \Pi_{mn}.$$

Then

$$\sum_{n\in\mathbb{Z}} [n]\, \mathrm{Tr}_{\mathcal{H}^{(0)}_{mn}} \tilde{\Pi}_m(a) X = \sum_{n=1}^{\infty} [n] \frac{\sin\frac{\pi a n}{r}}{\sin\frac{\pi n}{r}} (\mathrm{Tr}_{\mathcal{H}^{(0)}_{mn}} \Pi_{mn} X - \mathrm{Tr}_{\mathcal{H}^{(0)}_{m,-n}} \Pi_{m,-n} X)$$

$$= \sum_{n=1}^{\infty} [n]\, \mathrm{Tr}_{\mathcal{H}^{(1)}_{mn}} \tilde{\Pi}_m(a) X.$$

The operator $\tilde{\Pi}_m(a)$ is **invariant** with respect to the restriction.
For

$$a = a_\nu \equiv \frac{\nu}{q-p} \qquad \left(r = \frac{q}{q-p} \right)$$

the operator $\Pi_m(a)$ is invariant with respect to the restriction 2.
For $q = p + 1 = r$ (**unitary models**) we can identify

$$\tilde{\Pi}_m(a_\nu) \sim \phi_{\nu\nu}(x).$$

In particular for $r = 4$ (Ising model) the operator $\tilde{\Pi}_m(a_2)$ is the spin operator.

Scaling for form factors

To obtain the scaling form factors (**limit I**) we must substitute

$$\epsilon k \longrightarrow t$$

$$\sum_k \longrightarrow \int_{-\infty}^{\infty} dt$$

$$a_k \longrightarrow a(t)$$

But we need to take care with the summation over n in the zero modes. Namely, we have a sum like

$$\sum_n \mathrm{e}^{\mathrm{i}\pi a n/r} \mathrm{e}^{-2\epsilon P_{mn}^2 - \mathrm{i}\epsilon\sqrt{\frac{2r}{r-1}} P_{mn}\Theta/\pi},$$

$$\Theta = \sum_{i=1}^{N} \theta_i - 2\sum_{j=1}^{N/2} \gamma_j$$

Taking this sum we obtain the factor

$$P_m(a)\mathrm{e}^{-a\Theta/2(r-1)}.$$

The first factor $P_m(a)$ is the overall factor.

The second factor $\mathrm{e}^{-a\Theta/2(r-1)}$ modifies the integration in γ_j and maybe incorporated into the definition of the vertex operators as:

$$\Psi^*(a;\theta)_m^{m+1} = \mathrm{e}^{\mathrm{i}\varphi(\theta)}\boxed{\mathrm{e}^{-a\theta/2(r-1)}},$$

$$\Psi^*(a;\theta)_m^{m-1} = \eta^{-1}{:}\mathrm{e}^{\mathrm{i}\varphi(\theta)}{:}\boxed{\mathrm{e}^{-a\theta/2(r-1)}}\int_C \frac{d\gamma}{2\pi}{:}\mathrm{e}^{-\mathrm{i}\varphi(\theta+\mathrm{i}\pi)-\mathrm{i}\varphi(\theta-\mathrm{i}\pi)}{:}\boxed{\mathrm{e}^{a\gamma/(r-1)}}F_m(\gamma-\theta),$$

$$F_m(\gamma) = \frac{\mathrm{sh}\,\frac{\gamma+\mathrm{i}\pi/2-\mathrm{i}\pi m}{r-1}}{\mathrm{sh}\,\frac{\gamma-\mathrm{i}\pi/2}{r-1}}.$$

This is a new set of form factors for the perturbed minimal models.

In the **limit II**$_\pm$ it reduces to the known form factors for exponential operators of the sine-Gordon model, obtained by Lukyanov.