Smooth dynamical (de)-phantomization of a scalar field in simple cosmological models

A. Andrianov, F. Cannata, A. Kamenshchik, 2005,
Phys. Rev. D 72, 043531 (2005).

Some observations not only confirm the phenomenon of cosmic acceleration, but point out on an opportunity of the existence of the phantom dark energy, i.e. the matter with the relation between the pressure and dark energy

$$k \equiv \frac{p}{\rho} < -1$$

The evolution of the Universe filled with this type of matter can end in the cosmological singularity, called Big Rip.

$$\lim_{t \to t_R} a(t) \to \infty,$$

$$\lim_{t \to t_R} \dot{a}(t) \to \infty,$$

$$\lim_{t \to t_R} h(t) = \frac{\dot{a}(t)}{a(t)} \to \infty.$$

The moment of time when the parameter k changes its value from $k > -1$ to $k < -1$ is called
Phantom divide line.
There is a common wisdom that the crossing of the phantom divide line cannot be explained by simple scalar field models, because transition from non-phantom to phantom equation of state corresponds to change of sign of the kinetic term of the scalalr field:
non-phantom

$$L = \frac{\dot{\phi}^2}{2} - V(\phi),$$

phantom

$$L = -\frac{\dot{\phi}^2}{2} - V(\phi).$$

We argue that a smooth dynamical (de)-phantomization of a scalar field in simple cosmological models is possible.

Toy model with a crossing of the phantom divide line

$$ds^2 = dt^2 - a^2(t)dl^2,$$
$$p = k\rho,$$
$$h^2 = \rho,$$
$$\dot{\rho} = -3h(\rho + p),$$
$$a = a_0 t^{\frac{2}{3(1+k)}},$$
$$h = \frac{2}{3(1+k)t},$$

$$\dot{h} = -\frac{3}{2}(\rho + p),$$
$$p = -\frac{2}{3}\dot{h} - h^2.$$

For the scalar field

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi),$$
$$p = \frac{\dot{\phi}^2}{2} - V(\phi),$$
$$\dot{\phi}^2 = (\rho + p) = -\frac{2}{3}\dot{h},$$
$$V(\phi) = \frac{1}{2}(\rho - p) = \frac{\dot{h}}{3} + h^2.$$

The power-law evolution could be reproduced in the cosmological model with an exponential potential

$$V(\phi) \sim \exp\left(-\frac{3\sqrt{1+k\phi}}{2}\right).$$

The potential could be treated not as a function of the the value of the field ϕ, but as the function of time t.

S.Chervon, V.Zhuravlev, 1999, A.Yurov,2003

Then the volume function

$$\psi(t) \equiv a^3(t)$$

satisfies a simple second-order differential equation

$$\ddot{\psi} = 9V(t)\psi,$$

$$V(t) = \frac{2(1-k)}{9(1+k)^2 t^2}.$$

The general solution

$$\psi(t) = \psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2},$$

$$\alpha_1 = \frac{2}{1+k},$$

$$\alpha_2 = \frac{k-1}{1+k},$$

where ψ_1 and ψ_2 are nonnegative constants. One can derive the equation for the functions $t(\phi)$ compatible with the potential $V(t)$:

$$\left(\frac{1}{t'}\right)'' - \frac{9}{2t'} - \frac{4(1-k)}{9(1+k)^2}\left(\frac{t'^2}{t^3}\right)' = 0,$$

with the "prime" denoting the differentiation with respect to the scalar field ϕ. Particular solutions of this equation being generate different potentials $V(\phi)$, corresponding to different cosmological evolutions.
It is easy to check that the solution

$$t = \exp\frac{3\sqrt{1+k}\phi}{2},$$

corresponding to the exponential potential and to the evolution with $\psi_2 = 0$, satisfies the equation.
Instead of looking for other exact solutions

we concentrate now on the qualitative study of the space of potentials and solutions. The form of the Hubble variable corresponding to the evolution is

$$h = \frac{\alpha_1\psi_1 t^{\alpha_1 - 1} + \alpha_2\psi_2 t^{\alpha_2 - 1}}{3(\psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2})},$$

while its time derivative is

$$\dot{h} = -\frac{\alpha_1\psi_1^2 t^{2\alpha_1} + \alpha_2\psi_2^2 t^{2\alpha_2} + (1 - (\alpha_1 - \alpha_2)^2)\psi_1\psi_2 t^{\alpha_1+\alpha_2}}{3t^2(\psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2})^2}.$$

At large values of t, $\dot{h}$ written down up to the first non-leading term reads:

$$\dot{h} \approx -\frac{2}{3(1+k)t^2}\left(1 + \frac{2(k-3)\psi_2}{(1+k)\psi_1} t^{\frac{k-3}{1+k}}\right),$$

while the potential

$$V(\phi) \sim \exp\left(\frac{-3\sqrt{1+k}\phi}{2}\right)$$

$$\times\left(1 + 3\frac{\psi_2}{\psi_1}\frac{k-3}{\sqrt{1+k}}\exp\frac{3(k-3)\phi}{2\sqrt{1+k}}\right).$$

If one consider the cosmological evolution with non-zero value of the parameter ψ_2, one can state that there exists a potential describing this evolution.
At some moment t_0, such as $0 < t_0 < \infty$, the time derivative of the Hubble variable vanishes.

$$t_0 = \left(\frac{(3-k)\sqrt{2(1-k)}+4(1-k)}{2(1+k)}\frac{\psi_2}{\psi_1}\right)^{\frac{1+k}{3-k}}.$$

At $t > t_0$ the evolution is described by the non-phantom scalar field Lagrangian.
It is possible to find the approximate expression for the potential $V(\phi)$ for $t \to t_0+$.

$$\dot{h} = -H(t-t_0),$$

where

$$H = \frac{\sqrt{8(1-k)}(3-k)^2\psi_1\psi_2 t_0^{-\frac{4k}{1+k}}}{(1+k)^3\left(\psi_1 t_0^{\frac{2}{1+k}} + \psi_2 t_0^{\frac{k-1}{k+1}}\right)^2}.$$

$$\phi \approx \sqrt{\frac{8H}{27}}(t-t_0)^{3/2},$$

$$t = t_0 + \left(\sqrt{\frac{27}{8H}}\phi\right)^{2/3}.$$

Thus, the potential results

$$V(\phi) = \frac{2(1-k)}{9(1+k)^2\left(t_0 + \left(\sqrt{\frac{27}{8H}}\phi\right)^{2/3}\right)^2}.$$

It is rather straightforward to verify that the expressions for $\phi, \dot{\phi}$ and the potential $V(\phi)$ are regular when $t \to t_0+$ while the derivative $\frac{dV(\phi)}{d\phi}$ and $\ddot{\phi}$ are singular.

Another branch of the evolution

$t < t_0$.

For small values of t one should use the phantom Lagrangian. One can find the corresponding potential of the phantom field in the subleading approximation:

$$V(\phi) = \frac{2(1-k)}{9(1+k)^2}\exp\left(-3\sqrt{\frac{2(1+k)}{1-k}}\phi\right)$$

$$\times\left((1 + \frac{4\psi_1(1+k)}{\psi_2(3-k)}\exp\left(3\phi(3-k)\sqrt{\frac{1}{2(1-k^2}}\right)\right)$$

The possibility of matching these two branches of the evolution at $t < t_0$ and $t > t_0$.

Are they really incompatible ?

Let us consider a scalar field model with a negative kinetic term and a potential whose asymptotic form is given above. Suppose that at some moment $t < t_0$ one has initial conditions on the values of h, ϕ and $\dot{\phi}$

with $\psi_2 > 0$. Then approaching the time moment $t \to t_0-$ one arrives to the regime when $\dot{h}, \phi$ and $\dot{\phi}$ tend to vanish. Nevertheless, all the geometric characteristics of the spacetime remain well defined. In contradistinction, the second time derivative of the scalar field at $t = t_0$ and $\frac{dV(\phi)}{d\phi}$ diverge, but this divergence is an integrable one. This offers us an opportunity (and, perhaps, necessity) of a continuation of the spacetime geometry and field configurations beyound this "divide line". Clearly, such a smooth dynamical continuation which respects Einstein equations, entails the change of the sign of the kinetic term and the transition to the regime for $t > t_0$.

Different regimes of crossing of phantom divide line

The presented cosmological evolution describes the following scenario: the universe begins its evolution from the cosmological singularity of the "anti-Big Rip" type and its squeezing is driven by a scalar field with a negative kinetic term. Then at the moment

$$t_1 = \left(\frac{(1-k)\psi_1}{2\psi_2}\right)^{\frac{3-k}{1+k}}$$

the contraction of the universe is replaced by an expansion. At the moment $t = t_0 > t_1$ the kinetic term of the scalar field changes sign. Then, with the time growing the universe undergoes an infinite power-law expansion. From observational point of view, it would be more interesting to get an evolution beginning from the Big Bang and ending in the Big Rip singularity, after un-

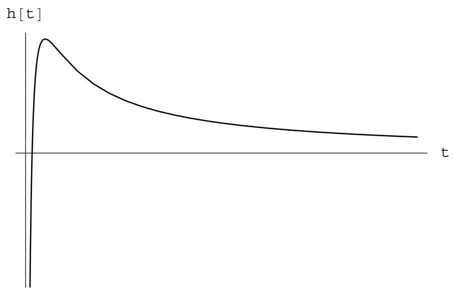

Figure 1: $h(t)$ dependence in our toy model

dergoing a phantomization transition. Instead of trying to construct some potential and cosmological evolution describing such a process we shall give a graphical presentation of the $h(t)$ -dependence which could be responsible for such a scenario. Some obser-

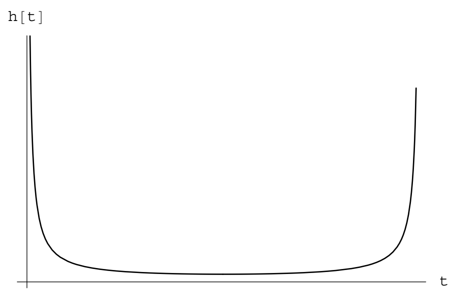

Figure 2: $h(t)$ dependence in the model, describing the Big Rip.

vations favor double crossing of the phantom divide line. To this time dependence

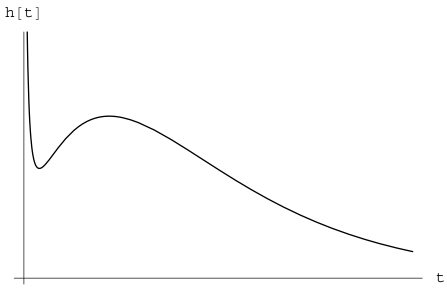

Figure 3: $h(t)$ dependence in the model, describing a double crossing of the phantom divide line.

of $h(t)$ one can associate some potential and the evolution, in course of which the scalar field undergoes two smooth transitions: phantomization at the point of mimimum of the curve $h(t)$ and the subsequent dephantomization at the point of its maximum.

This diagram reminds topologically the well-known van der Waals isoterm curve. The part of the van der Waals curve situated between the maximum and minimum, describing a metastable state (of supercool vapor or superheated liquid), is analogous to the phantom phase.

Discussion

We were guided by a belief that Einstein equations are more fundamental then the concrete form of the action for other fields. The idea about the dominant role of Einstein equations goes back to the classical works by Einstein, Infeld and Hoffmann , where it was shown that the motions of mass points (geodesics law) are determined by the Einstein equations for the gravitational field.

The derivability of the equation of motion for particles is based on the fact that the Einstein equations are non-linear and also they are subject to additional identities, namely the Bianchi identities which reduce the number of independent equations. These Bianchi identities imply conservation laws for matter, present in the model under consideration.

The argumentation, based on the Bianchi identities could be used also for getting information about the possible field configurations for non-gravitational fields.

The Einstein equations

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = \kappa T^{\mu}_{\nu}.$$

The Bianchi identities

$$\nabla_{\mu}\left(R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R\right) = 0$$

imply some kind of the energy-momentum conservation law:

$$\nabla_{\mu} T^{\mu}_{\nu} = 0.$$

For the case of the minimally coupled spatially homogeneous time-dependent scalar field with the standard sign of the kinetic term

$$\left(\ddot{\phi} + 3h\dot{\phi} + \frac{dV}{d\phi}\right)\dot{\phi} = 0,$$

which is equivalent to the Klein-Gordon equation when $\dot{\phi} \neq 0$. For the phantom scalar field

$$\left(-\ddot{\phi} - 3h\dot{\phi} + \frac{dV}{d\phi}\right)\dot{\phi} = 0.$$

The comparison between these equations points to the opportunity of the change of type of the Klein-Gordon equation at the moment when $\dot{\phi} = 0$. Thus, while the Einstein equations require the change of the type of the scalar field Lagrangian provided some form of the scalar field potential and the cosmological evolution are chosen, the Bianchi identities show why this transformation is possible.

Question

Given the potential as a function of the scalar field (not of the time) how general are the initial conditions providing the reaching of the point $\dot{\phi} = 0$? Do we need the fine tuning?

Phantom universe from CPT symmetric QFT

A. Andrianov, F. Cannata, A. Kamenshchik, 2005, gr-qc/0512038

We present a rather simple and natural cosmological toy model, linked to such an intensively developing branch of quantum mechanics and quantum field theory as the study of non-Hermitian, but CPT (or PT) symmetric models C.M. Bender, S. Boettcher, 1998.

There exists a large class of non-Hermitial Hamiltonians, which nevertheless possesses real and often positive definite spectrum. They are characterized by a potential which in one-dimensional case satisfies the property of PT - invariance $V(x) = V^*(-x)$.

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* - V(\phi,\phi^*), \quad (1)$$

with a potential $V(\phi,\phi^*)$ satisfying the CPT invariance condition

$$(V(\phi, \phi^*))^* = V(\phi^*, \phi), \tag{2}$$

while the condition

$$(V(\phi, \phi^*))^* = V(\phi, \phi^*), \tag{3}$$

is not satisfied.

Such potential can have a form

$$V(\phi, \phi^*) = V_1(\phi + \phi^*)V_2(\phi - \phi^*). \tag{4}$$

Defining

$$\phi = \phi_1 + i\phi_2 \tag{5}$$

we consider the potentials of the form

$$V(\phi, \phi^*) = V_0(\phi_1)\exp(i\alpha\phi_2). \tag{6}$$

The functions ϕ_1 and ϕ_2, introduced as the real and imaginary parts of the complex scalalr field, will be treated as independent spatially homogeneous variables depending only on the time parameter t.

$$\ddot{\phi}_1 + 3h\dot{\phi}_1 + V_0'(\phi_1)\exp(i\alpha\phi_2) = 0, \tag{7}$$

$$i\ddot{\phi}_2 + 3ih\dot{\phi}_2 - \alpha V_0(\phi_1)\exp(i\alpha\phi_2) = 0, \tag{8}$$

$$h \equiv \frac{\dot{a}}{a}, \tag{9}$$

$$ds^2 = dt^2 - a^2(t)dl^2, \tag{10}$$

$$h^2 = \frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + V_0(\phi_1)\exp(i\alpha\phi_2). \tag{11}$$

The system of equations (7),(8),(11) can have a solution where $\phi_1(t)$ is real, while ϕ_2 is imaginary

$$\phi_2(t) = -i\xi(t), \tag{12}$$

$\xi(t)$ is real.

$$\ddot{\phi}_1 + 3\sqrt{\frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1)\exp(\alpha\xi)}\,\dot{\phi}_1 + V_0'(\phi_1)\exp(\alpha\xi) = 0, \tag{13}$$

$$\ddot{\xi} + 3\sqrt{\frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1)\exp(\alpha\xi)}\dot{\xi}$$
$$-\alpha V_0(\phi_1)\exp(\alpha\xi) = 0. \tag{14}$$

Energy density

$$\varepsilon = h^2 = \frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1)\exp(\alpha\xi). \tag{15}$$

Pressure

$$p = \frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 - V_0(\phi_1)\exp(\alpha\xi). \tag{16}$$

If $\dot{\phi}_1^2 < \xi^2$ the pressure will be negative and $p/\varepsilon < -1$, satisfying the phantom equation of state.
The condition

$$\dot{\phi}_1^2 = \dot{\xi}^2 \tag{17}$$

corresponds exactly to the phantom divide line.

The exactly solvable cosmological model.

The Hubble variable:

$$h(t) = \frac{A}{t(t_R - t)}. \tag{18}$$

$t = 0$ - a standard initial Big Bang cosmological singularity,
$t = t_R$ - a Big Rip.
The derivative of the Hubble variable

$$\dot{h} = \frac{A(2t - t_R)}{t^2(t_R - t)^2} \tag{19}$$

vanishes at

$$t_P = \frac{t_R}{2} \tag{20}$$

when the universe crosses the phantom divide line.

Standard technique of reconstruction of the potential;

$$\frac{\dot{\phi}_1^2}{2} - \frac{\dot{\xi}^2}{2} + V_0(\phi_1)e^{\alpha\xi} = h^2 = \frac{A^2}{t^2(t_R - t)^2}, \tag{21}$$

$$\frac{\dot{\phi}_1^2}{2}-\frac{\dot{\xi}^2}{2}-V_0(\phi_1)e^{\alpha\xi} = -\frac{2}{3}\dot{h}-h^2 = -\frac{A(4t-2t_R+3A)}{3t^2(t_R-t)^2}. \tag{22}$$

$$V_0(\phi_1) = \frac{A(2t-t_R+3A)}{3t^2(t_R-t)^2}e^{-\alpha\xi}. \tag{23}$$

$$\dot{\phi}_1^2 - \dot{\xi}^2 = -\frac{2A(2t-t_R)}{3t^2(t_R-t)^2}. \tag{24}$$

$$\ddot{\xi} + 3\dot{\xi}\frac{A}{t(t_R-t)} - \frac{\alpha A(2t-t_R+3A)}{3t^2(t_R-t)^2} = 0. \tag{25}$$

$$m \equiv \frac{3A}{t_R}. \tag{26}$$

The equation of state parameter w in the vicinity of the initial Big Bang singularity behaves as

$$w = -1 + \frac{2}{m}, \tag{27}$$

while approaching the final Big Rip singularity this parameter behaves as

$$w = -1 - \frac{2}{m}. \tag{28}$$

Results:

$$\xi = \frac{\alpha m}{9}(\log t - \log(t_R - t)). \qquad (29)$$

For the case $\alpha^2 m = 18$

$$\phi_1 = \sqrt{32}\mathrm{Arctanh}\sqrt{\frac{t_R - t}{t_R}}. \qquad (30)$$

$$t = \frac{t_R}{\cosh^2 \frac{\phi_1}{\sqrt{32}}}. \qquad (31)$$

The explicit expression for the potential $V_0(\phi_1)$:

$$V_0(\phi_1) = \frac{2\cosh^6 \frac{\phi_1}{\sqrt{32}}\left(2 + 17\cosh^2 \frac{\phi_1}{\sqrt{32}}\right)}{t_R^2}. \qquad (32)$$

At the moment $t_P = t_R/2$, the equality $\dot{\xi}^2 = \dot{\phi}_1^2$ is satisfied and the universe is crossing the phantom divide line.

The potential (32) is smooth together with all its derivatives at this point.

Conclusion

We have constructed a model relaxing the requirement of Hermiticity of the Hamiltonian of the theory which is equivalent to the reality of the classical Lagrangian. For our classical solutions, expressed in terms of real fields, observable quantities like energy density, pressure, Hubble variable turn out to be real. As a consequence our model describes in a rather natural way the transition from normal matter to phantom one.