Free energy distribution of the (1+1)-dimensional directed polymer problem

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Outline

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4. Saving replicas
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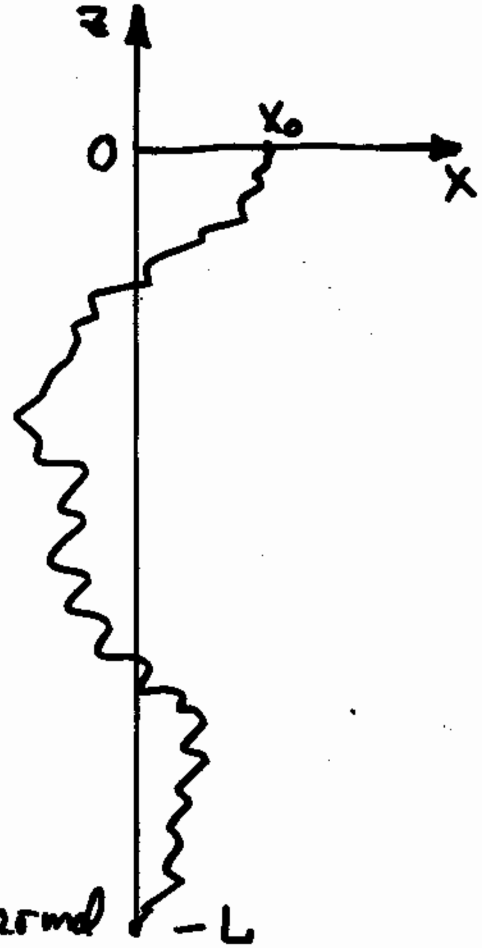
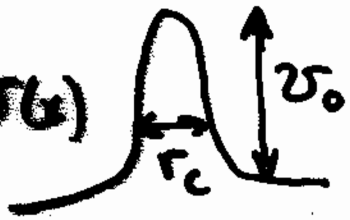
$$H[x(z)] = \int_{-L}^{0} dz \left\{ \frac{c}{2}\left(\frac{\partial x}{\partial z}\right)^2 + V\big(x(z), z\big) \right\}$$



with correlator

$$\overline{V(x,z)\, V(x',z')} = \delta(z-z')\, \upsilon(x-x')$$

$\upsilon(x)$ 

$$\upsilon(x) = u\,\delta(x)$$

displacement $\overline{\langle X^2(L)\rangle} \equiv \frac{1}{L}\int_{-L}^{0} dz\, \overline{\langle x^2(z)\rangle}$

$$\overline{\langle x^2(L)\rangle} \propto L^{2\xi}$$

where wandering exponent

$$\xi_{1+1} = \frac{2}{3}\ , \quad \xi_{1+2} \simeq 0.620$$

$\langle\ \rangle$ - thermal

$\overline{(\ )}$ - disorder

Partition function

$$Z(L, x_0; V) = \int_{x(-L)=0}^{x(0)=x_0} \mathcal{D}x(z) \exp[-\beta H(x, V)]$$

free energy $\quad F(L, x_0; V) = -T \ln Z$

Scaling of energy fluctuations as function of L

$$\delta F(L) \propto L^{\omega}$$

equating it to elastic energy

$$\delta F \sim c \frac{x^2}{L} \sim L^{\omega} \Rightarrow x^2 \propto L^{\omega+1} \Rightarrow \zeta = \frac{\omega+1}{2}$$

If we know $\quad \delta F \propto x_0^{\nu} \Rightarrow$

$$\delta F \sim \frac{c x_0^2}{L} \sim x_0^{\nu} \Rightarrow \zeta = \frac{1}{2-\nu}$$

Mapping to Burgers turbulence

For fixed $V(x,z)$ partition function satisfies the imaginary time Schrödinger equation

$$\frac{\partial Z}{\partial \tau} = \frac{1}{2\beta c}\frac{\partial^2 Z}{\partial x^2} - \beta V(x,\tau) Z$$

introducing free energy $h = \frac{F}{c} = -\frac{\ln Z}{\beta c}$ we obtain KPZ eq.

$$\frac{\partial h}{\partial \tau} + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{1}{2\beta c}\frac{\partial^2 h}{\partial x^2} = \frac{V(x,\tau)}{c}$$

going over to the velocity field $v(x) = \frac{\partial h}{\partial x}$

we find the randomly driven Burgers equation

$$\frac{\partial v}{\partial \tau} + \left(v\frac{\partial}{\partial x}\right)v - \frac{1}{2\beta c}\frac{\partial^2 v}{\partial x^2} = \frac{1}{c}\frac{\partial V(x,\tau)}{\partial x}$$

viscosity η $\Rightarrow$ our $T \to 0$ corresponds to $\eta \to 0 \Rightarrow$ developed turbulence

Replica approach

$$Z[L,V] = \exp(-\beta\, F[L,V])$$

replicating partition function and averaging over disorder

$$Z(n,L) \equiv \overline{Z^n(L,V)} = \overline{\exp(-\beta n\, F[L,V])}$$

since $F(L,V)$ is itself a random quantity, denoting its distribution function by $P_L(F)$ and replacing $\beta n \equiv \xi$

$$Z(\xi,L) = \int_{-\infty}^{\infty} dF\, P_L(F) \exp(-\xi F)$$

Thus taking inverse Laplace transform of $Z(n,L)$ we obtain $P_L(F) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\xi\, Z(\xi,L) \exp(\xi F)$

and the moments $\langle \overline{F^K(L)} \rangle = (-1)^K \left. \frac{\partial^K Z(\xi,L)}{\partial \xi^K} \right|_{\xi=0}$

Kardar's solution (1987)

Replicating our Hamiltonian and performing disorder averaging we obtain for the partition function

$$Z = \prod_{a=1}^{p} \int_{x_a(-L)=0}^{x(\tau)=x_a} \mathcal{D}[x_a(z)] \exp\left[-\beta H_n(x_1(z) \ldots x_p(z))\right]$$

with the replica Hamiltonian

$$H_n = -\frac{1}{2\beta c} \sum_{1}^{p} \frac{\partial^2}{\partial x_a^2} - \frac{\beta^2}{2} \sum_{a,b=1}^{p} \mho(x_a(z) - x_b(z))$$

Z has the meaning of the wave function, that satisfies imaginary-time Schrödinger Eq.

$$-\frac{\partial \psi(x,\tau)}{\partial \tau} = H_n \psi(x,\tau)$$

with initial condition $\psi(x,-L) = \delta^p(x)$

M. Kardar (1987) argued, that for long time limit L solution is dominated by the ground state E_0 and

$$Z \propto \exp(-\beta E_0 L)$$

For $\mathcal{V}(x) = u\,\delta(x)$ he solved this Schrödinger Eq. by Bethe ansatz and obtained

$$\beta E_0 = -\frac{\beta^2}{2}\mathcal{V}(0)\,p - \frac{\beta^5 c u^2}{6}(p^3 - p)$$

Since $\overline{\langle F^k(L)\rangle} = (-1)^k \left.\frac{\partial^k Z(\xi, L)}{\partial \xi^k}\right|_{\xi = \beta p = 0}$

one obtains $\boxed{\overline{\langle F^3(L)\rangle} = \beta^2 c u^2 L}$

- $\Rightarrow \delta F \sim L^{1/3} \Rightarrow \omega = \frac{1}{3} \Rightarrow x^2 \propto L^{2\xi}$

with $\boxed{\xi = \frac{2}{3}}$

Problems

Since in this solution $E_0 = e_1 p + e_2 p^3$ then we obtain that only the first and the third moments of F^k are present, e.g. $\overline{\langle F^2 \rangle} = 0 \Longrightarrow$ pathology, meaning that $P_L(F)$ is negative somewhere.

What to do?

1. Ignore the problem and extract everything from $\overline{\langle F^3 \rangle}$
2. Trash this solution, since it leads to pathological $P_L(F)$
3. Try to understand why this difficulty appears and reformulate the problem in order to avoid it.

<u>Source</u> of the <u>problem</u>

As soon as we left only the ground state

$$Z(p,L) = \Psi(0,0) = \exp(-\beta E_0(p) L) \quad \text{we are done!}$$

Assuming the ground state energy $E_0(p) = e_0 + e_1 p + e_2 p^2 + e_3 p^3 + \dots$ there are only two possibilities:

if $e_2 \neq 0 \Rightarrow$ we have trivial random walk like scaling $\delta F(L) \propto L^{1/2}$;

if $e_2 = 0$ we have vanishing of the second moment and negative distribution function $P_L(F)$

Thus we even don't need to calculate $E_0(p)$ since we will get either trivial or wrong result

What got wrong?

In Kardar's solution $Z(p,L) \simeq \exp[-\beta E_0(p) L]$

distribution function $P_L(F) \sim \int_0^\infty \cos(\xi F + e_3 \xi^3 L/\beta^2)\, d\xi$

It gives the body of distribution function

at $F^3 \sim \dfrac{L e_3}{\beta^2}$

Typical ξ_L, that contribute to this integral are $\xi_L \sim \left(\dfrac{\beta^2}{L e_3}\right)^{1/3}$. Substituting this back to

$$Z(p,L) = \exp(-\underbrace{\beta E_0(p)|L}_{\sim 1}) \sim \exp(-1)$$

$\beta p \sim \xi_L$

but contribution of excited states is of the same order $\Rightarrow$ they cannot be ignored!

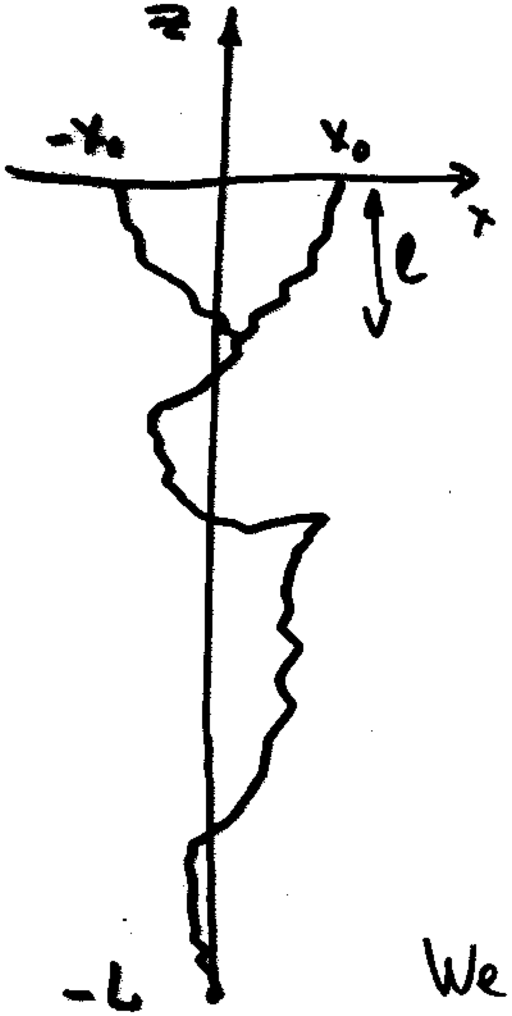
How to resque replicas?

On turbulence language

$P_L(F)$ corresponds to the different time corr. function $P_\tau(v)$, that is usually very hard to compute.

However, for equal time distribution function $P_x(v)$ there are some results.

So let us try to calculate it.



Let us calculate the free energy difference

$$F'(L,x_0,V) = F(L,x_0,V) - F(L,-x_0,V)$$

It corresponds to the partition function

$$Z'(x_0,V) = Z(x_0,V)\, Z^{-1}(-x_0,V)$$

and replicating

$$Z'(n,x_0) = \overline{Z^n(x_0,V)\, Z^{-n}(-x_0,V)} = \overline{\exp(-\xi F')}$$

Then $$P_{x_0}(F) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, Z'(\xi,x_0)\, e^{\xi F'}$$

We consider more general $Z'(n,m,x_0) \equiv \overline{Z^n(x_0,V) Z^m(-x_0,V)}$

$$= \Big[\prod_1^n \int_0^{x_0} \mathcal{D}x_a(z)\Big]\Big[\prod_1^m \int_0^{-x_0} \mathcal{D}x_b(z)\, \exp(-\beta H_{n+m}(x_1,x_2\ldots x_{n+m})\Big)$$

and take the limit $m=-n$, $Z'(n,x_0) = Z'(n,m,x_0)\Big|_{m=-n}$

For the long time limit $L \to \infty$ initial conditions $\tau = -L$ becomes irrelevant and at $\tau = 0$ $\Psi(x,0) = \exp(-\beta E_0 L)\,\Psi_0(x)$

Ground state wave function $\Psi_0\{x_i\} = \exp\left[-\frac{\beta^3 c u}{4} \sum_{a,b=1}^{n+m} |x_a - x_b|\right]$

We first take thermodynamic limit $L \to \infty$ and then $n+m=0$. As a result we obtain

$$\Psi(x_0, 0) = \Psi_0(x_0)$$

$$\Psi_0(x_0) = \exp\left[-\beta^3 c u\, m n\, |x_0|\right]$$

and after $m = -n$ $\Rightarrow$ $Z'(n, x_0) = \exp\left[\beta u c (\beta n)^2 |x_0|\right]$

Inverting it $P_{x_0}(F') = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \exp(-\beta u c |x_0| \xi^2 + i \xi F') =$

$$= \frac{1}{\sqrt{4\pi \beta u c |x_0|}} \exp\left(-\frac{F'^2}{4\beta u c |x_0|}\right)$$

Thus we have Gaussian distribution functions

$$\delta F' \equiv \overline{\langle F'^2 \rangle}^{1/2} = \sqrt{2\beta u c}\; x_0^{1/2}$$

Comparing it to elastic term $\frac{c\, x_0^2}{\ell}$

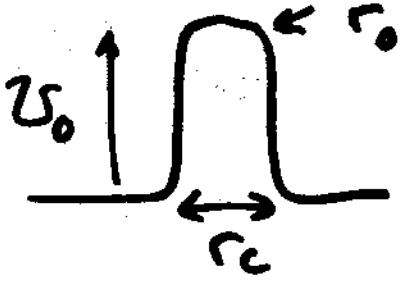
we obtain $x(\ell) \sim \left(\frac{\beta u}{c}\right)^{1/3} \ell^{2/3}$

Note, that the Gaussian distribution function obtained above can be checked via mapping to Burgers turbulence, where the steady state distribution functions for velocity field is

$$P_B(v(x)) \propto \exp\left\{-\frac{c}{2\beta u}\int_{-x_0}^{x_0} dx\, v^2(x)\right\}$$

$$F'(x_0) = c\int_{-x_0}^{x_0} dx\, v(x)$$

Generalizations, correlator of finite range



There is replica symmetry breaking – V. Dotsenko & S. Korshunov 98.
particles split into clusters of K particles, that are trapped within the bottom of parabolic well. Within these clusters interaction is quadratic, between the clusters it is $\propto \delta(r)$
As a result we obtain again the Gaussian distribution

$$P_{x_0}(F') = \sqrt{\frac{r_c}{4\pi F_r^2 |x_0|}} \exp\left(-\frac{F'^2}{4F_r^2 (|x_0|/r_c)}\right)$$

with $$F_r = \left(\frac{c\, v_0\, r_c^3}{\sqrt{2}\, r_0}\right)^{1/3}$$

$$\Rightarrow \delta F \sim F_r \left(\frac{x_0}{r_c}\right)^{1/2}$$

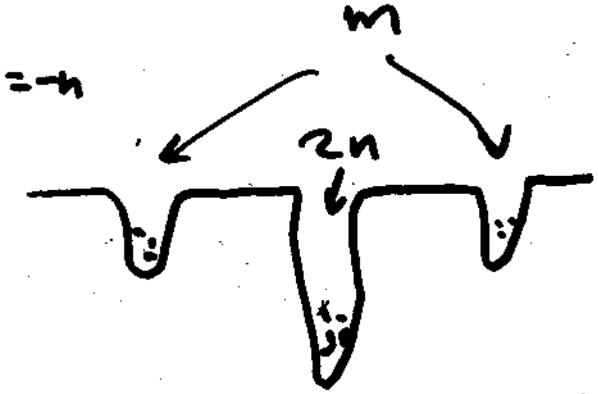
For large $F' > F_r \dfrac{|x_0|}{r_c}$

typical distance between the clusters becomes $\sim r_c$
and one should modify the calculations
In this situation we can proceed by looking at

$$F'' = 2F(0,V) - F(x_0,V) - F(-x_0,V) \quad \text{or}$$

$$Z'' = \overline{Z^{2n}(0,V)\, Z^{m}(x_0,V)\, Z^{m}(-x_0,V)}\,\Big|_{m=-n}$$

that corresponds to the three wells



Considering quasiclassical tunneling
between the wells we obtain $Z''(n,x_0) \sim \exp\left[2\sqrt{3c v_0}\,|x_0|(\beta n)^{3/2}\right]$

or $\quad P_{x_0}(F'') \sim \exp\left[-\dfrac{|F''|^3}{9c v_0 x_0^2}\right]$ M. Feigelman 198...

Instanton solution

KPZ equation for F is

$$\partial_\tau F + \frac{1}{2c}(\partial_x F)^2 - \frac{1}{2\beta c}\partial_x^2 F = V(x,\tau)$$

Using Martin-Siggia-Rose formalism we rewrite the constraint

$$\int \mathcal{D}V(x,\tau)\,\mathcal{D}F(x,\tau)\,P[V(x,\tau)]\,\delta\left(\dot F + \frac{F'^2}{2c} - \frac{F''}{2\beta c} - V\right) = 1$$ through

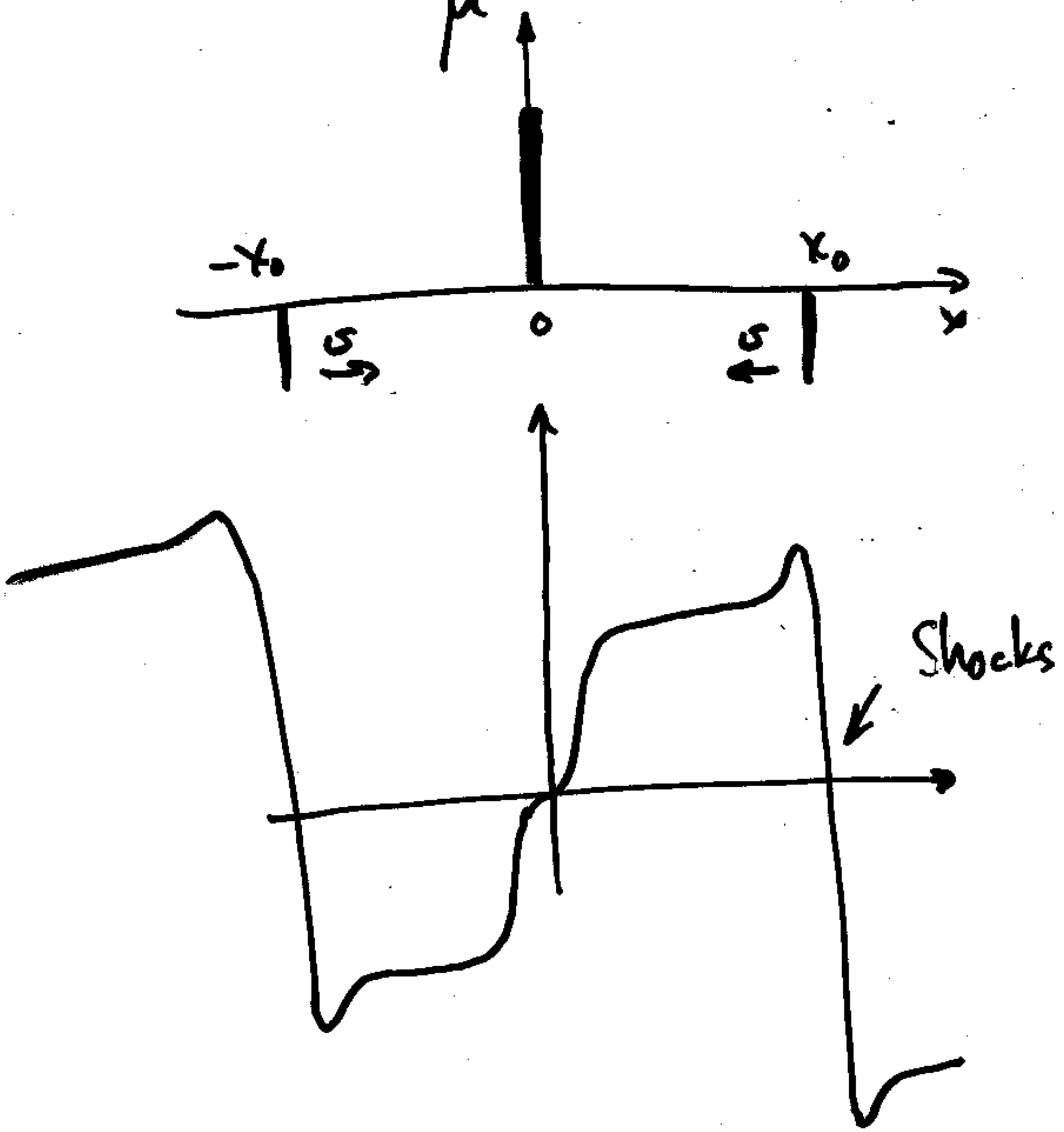
$$\delta[\ldots] = \int \mathcal{D}\mu(x,\tau)\,\exp\left(i\int_{-L}^{\tau} d\tau_1 \int_{-\infty}^{\infty} dx\,\mu(x,\tau_1)\,[\ldots]\right)$$

with effective action

$$S_{\xi,x_0}[F,\mu] = i\int_{-L}^{\tau} d\tau_1 \int_{-\infty}^{\infty} dx\,\mu(x,\tau_1)\left[\dot F + (\partial_x F)^2/2c - \frac{\partial_x^2 F}{2\beta c}\right] -$$

$$-\frac{1}{2}\int_{-L}^{\tau} d\tau_1 \int_{-\infty}^{\infty} dx\,dx'\,\mu(x,\tau_1)\,\mho(x-x')\,\mu(x',\tau_1) -$$

$$-\,\xi\int_{-L}^{\tau} d\tau_1 \int_{-\infty}^{\infty} dx\,F(x,\tau_1)\left[2\delta(x) - \delta(x-x_0) - \delta(x+x_0)\right]\delta(\tau_1-\tau)$$



Calculating evolution of μ, v we obtain

$$S_{\xi, x_0} = 2\sqrt{3}\, \xi^{3/2} \sqrt{c\, 2 v_0}\, |x_0|$$

in agreement with replica result