Non-superfluid Bose liquid at T = 0 in 2D

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1. Introduction: why it is interesting?

2. 2D Bosons + gauge field: origin and consequencies

3. Supersymmetric model of 2D long-range Bose-liq.
   - a. SUSY many-body QM
   - b. Matrix dynamics
   - c. Estimates for phase boundaries
   - d. Relation to classical Monte-Carlo
   - e. Spontaneous breakdown of Galilei invariance ? and formation of a "Bose-line" ?

"On the existence of Bose Metal at $T=0$" Das and Doniach PRB 1999

"Bosons in fluctuating gauge fields: Bose metal and phase separation"
R. Jack and D.K. Lee PRL 2002

"Ring exchange, Bose metal and bosonization in 2D"
A. Paramekanti, L. Balents, M.P. Fisher
PRB 2002

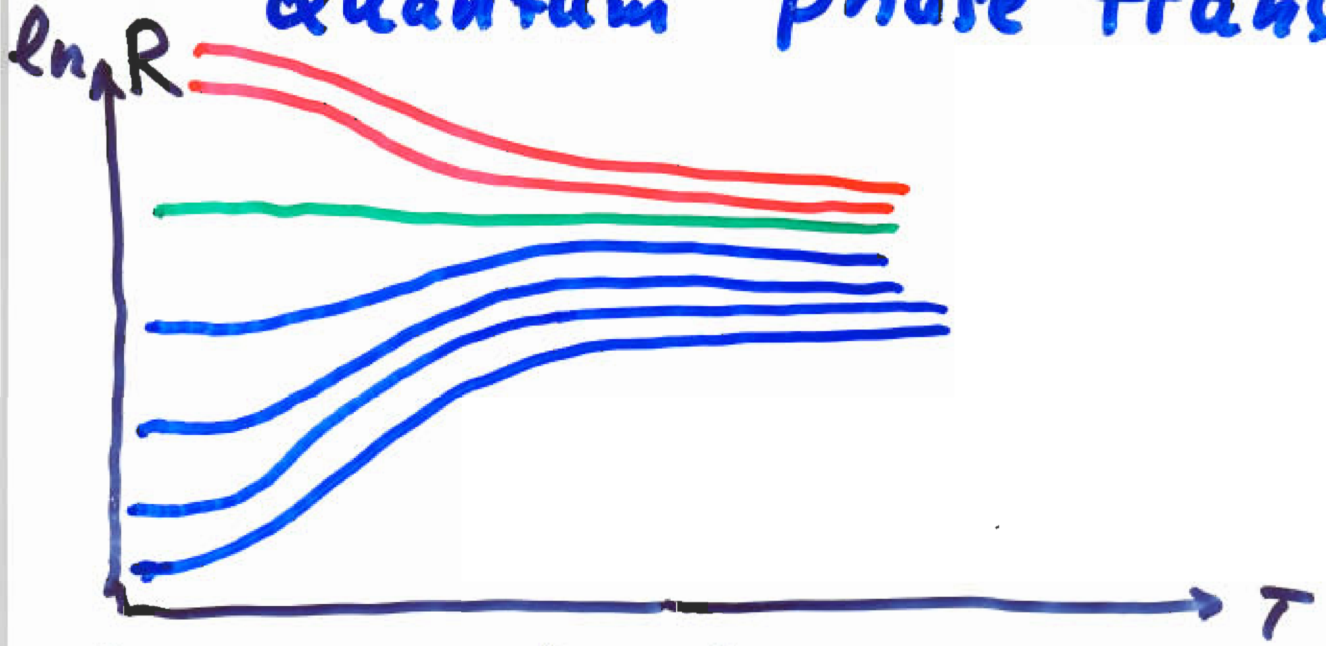
"The Elusive Bose metal"
P. Philips, D. Dalidovich 2003

"Do Bose metals exist in Nature?"
S. Sorella 2005

"Phase Diagram of the Bose-Hubbard model with T_3 symmetry"
M. Rizzi, V. Cataudella, R. Fazio
2005

Why is it interesting?

1. Exp. data near S-I Quantum phase transition



Low-T saturation of resistance at non-universal, strongly parameter-dependent values

Both artificial arrays and granular metals, and even „homogeneous" films

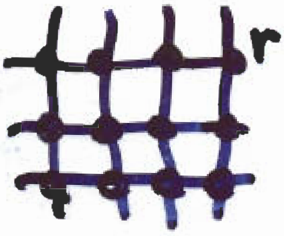
2. RVB-like approach to HTSC microscopics

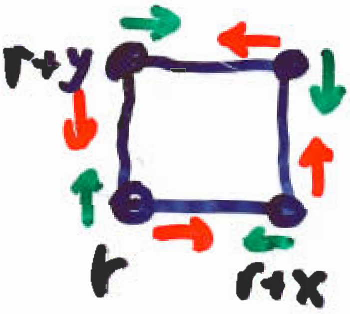
3. Classical statistics of fluctuating flux lines

4. "Composite boson" approach to FQHE

The only solved example of „Bose metal" in 2D: lattice ring exchange model, Paramekanti et al

$$H_\square = \frac{U}{2}\sum_{\bar r}(n_{\bar r}-\bar n)^2 - K\sum_{\bar r}\cos(\Delta_{xy}\phi_{\bar r})$$





$$\Delta_{xy}\phi_{\bar r} = \phi_{\bar r}-\phi_{\bar r+\bar x}-\phi_{\bar r+\bar y}+\phi_{\bar r+\bar x+\bar y}$$

$$[\phi_{\bar r}, n_{\bar r'}] = i\delta_{\bar r,\bar r'}$$

Main result: $E_{\bar k} = \text{const}\left|\sin\frac{k_x}{2}\sin\frac{k_y}{2}\right|$

gapless excitations with large phase volume!

no superfluidity

The most fundamental formulation: is it possible that 2D Bosons constitute non-SF Liquid in translation invariant system?

This question was addressed in:

① Feigelman, Geshkenbein, Vinokur JETP Lett. 1990

② Feigelman, Ioffe, Geshkenbein, Larkin PRB 1993

③ Feigelman, Skvortsov Nucl. Phys. B 1997

The answer is still unknown

F. G. I. L. PRB 1993

Why could SF state be destroyed at T=0 without lattice or disorder?

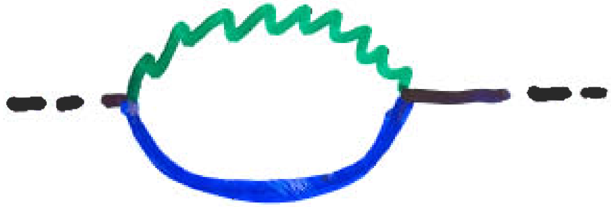
$$\mathcal{L} = \psi^*\left[i\left(\frac{\partial}{\partial t} - i a_0\right) + \frac{1}{2m}(\vec{\nabla} - i\vec{a})^2\right]\psi - V_{SR}(\psi^*\psi) + \frac{1}{2g^2} f_{0\alpha}^2 - \frac{c^2}{2g^2} f_{12}^2$$

$\alpha_c = \frac{m g^2}{16\pi^2 n}$ Scalar coupling

$\alpha_g = \frac{g^2}{8\pi m c^2}$ gauge coupling

Galilei inv. is broken if $\alpha_g \neq 0$

Correction to n_s:



$$1 - \frac{n_s}{n} = \alpha_g \, J(\alpha_c/\alpha_g)$$

n_s suppression is large at $\alpha_g \sim 1/2$

Conclusion #2 :

U(1) symmetry breaking is not necessary for superconductive response to exist !

Therefore we will try to establish relations between superconductive response and properties of the vortex system

Equilibrium classical thermodynamics of lines in 3D $\equiv$ Many-body quantum mechanics of 2D Bose liquid .

Superconductor	2D Bose liquid ($T^B = 0$)
Abrikosov lattice	Wigner-like crystal
Normal state	Superfluid
Vortex liquid	Non-SF liquid

(a phase with SC coherence along $\vec{B}_e$: $j_z = - n_s \delta A_z$)

Ⓐ. Vortex liquid : classical statistical mechanics.

L5

Type-II superconductor in the mixed state,

$B \ll H_{c2} \Rightarrow \underset{\substack{\uparrow \\ \text{distance} \\ \text{between VL.}}}{a_0} = \sqrt{\frac{\varphi_0}{B}} \gg \xi \leftarrow$ coherence length

Free energy of some configuration of vortex lines is (London approximation):

$$F = \frac{\varepsilon_0}{2} \sum_{ij} \iint d\vec{r}_j \, d\vec{r}_i \cdot \frac{\exp(-|\vec{r}_i - \vec{r}_j|/\lambda)}{|\vec{r}_i - \vec{r}_j|} - \frac{1}{4\pi} \int d^3x \, \underset{\substack{\uparrow \\ \text{external} \\ \text{field}}}{\vec{H}_{ext}} \cdot \vec{B}$$

λ - London penetration depth

$\varepsilon_0 = \left(\frac{\varphi_0}{4\pi\lambda}\right)^2 \qquad \phi_0 = \frac{hc}{2e}$

$$\underset{\substack{\nwarrow \\ \text{partition} \\ \text{function}}}{Z} = \int \mathcal{D}\vec{r}_j(z) \, e^{-F/T}$$

$$F\{\vec{r}_j, \underline{a}, \underline{A}\} = \sum_j \int dz \left[\frac{\varepsilon_0}{2} \left(\frac{d\vec{r}_i}{dz}\right)^2 - \mu \right] + \int d^3x \Big[i\Big(j_\alpha - \frac{1}{\varphi_0} \epsilon_{\alpha\beta\gamma} \nabla_\beta A_\gamma\Big) a_\alpha$$
$$+ \underset{\substack{\nwarrow \\ \text{kinetic} \\ \text{energy of supercurrent}}}{\frac{1}{4g^2} f_{\alpha\beta}^2} + \underset{\substack{\uparrow \\ \text{magnetic} \\ \text{energy}}}{\frac{1}{8\pi} (\underline{\nabla} \times \underline{A})^2} \Big]$$

$g^2 = 4\pi\varepsilon_0 = \frac{\varphi_0^2}{4\pi\lambda^2}$

$\mu = \frac{\varphi_0 H_{ext}}{4\pi} - \varepsilon_0 \ln\frac{\lambda}{\xi} \qquad f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$

$j_\alpha = (n, \vec{j}\,)$:

$n(\vec{r}, z) = \sum_j \delta(\vec{r} - \vec{r}_j(z)) \qquad \vec{j}(\vec{r}, z) = \sum_j \frac{d\vec{r}}{dz} \delta(\vec{r} - \vec{r}_j(z))$

Mapping to a 2D quantum bosons : (Nelson 88)

lines $\vec{r}_j(z)$ are considered as a Feynman's world lines of particles.

z	τ
T	$\hbar$
L_z	$\hbar / T_B \equiv \beta$

The integral over trajectories is transformed into the integral over complex Bose field :

$$S = \int \left[L_B(\psi, \underset{\sim}{a}) + L_G(A, a) \right] d^2 r \, d\tau$$

$$L_B = \psi^* \left[\frac{\partial}{\partial \tau} - \frac{1}{2m} (\bar{\nabla} - i \bar{a})^2 + i a_0 - \mu \right] \psi + V_{sr}(\psi^* \psi) + \\ + \frac{1}{4g_2} f_{\alpha\beta}^2$$

$$L_G = \frac{i}{4\sqrt{\pi} \lambda g} \varepsilon_{\alpha\beta\gamma} A_\alpha f_{\beta\gamma} + \frac{1}{8\pi} [\underline{\nabla} \times \underline{A}]^2$$

(imaginary time τ)

L_B is the Lagrangian we are going to study $\boxed{\alpha_g = 1/2}$

The influence of $\underline{A}$ fluctuations is to produce finite gauge-invariant mass $\frac{1}{\lambda}$ to the $\underset{\sim}{a}$ field

$\left(L_{eff}(\underset{\sim}{a}) = \frac{1}{4g^2} f_{\alpha\beta}^2 + \frac{1}{2g^2\lambda^2} a_\perp^2 \right)$.

In the limit $n\lambda^2 \gg 1$ (i.e. $B \gg H_{c1}$)

this effect is irrelevant for the determination of G.S.

$\underline{A}$ will serve as a source field to probe the ground-state properties.

"Normal" Bose-liquid ground state: implications

Vortex liquid in type-II superconductors:

$$\mathcal{L}_{eff}\{\underline{A},\underline{a}\} = \frac{i}{4\sqrt{\pi}\lambda g}\epsilon_{\alpha\beta\gamma}A_\alpha f_{\beta\gamma} + \frac{1}{8\pi}[\underline{\nabla}\times\underline{A}]^2 + \frac{1}{4g^2}f_{\alpha\beta}^2$$

$$+ \frac{1}{2}\Pi_\perp\cdot\vec{a}_\perp^2 + \mathcal{L}_{\parallel}\{a_0\}$$

transverse Bosonic polarisation function

$(\vec{\nabla}\vec{a} = 0)$.

$$\mathcal{D}_{zz}(\vec{q}) = \langle A_z(q)A_z(-q)\rangle = \frac{4\pi T}{q^2 + P_\perp(q)};$$

$$P_\perp(q) = \frac{1}{\lambda^2}\frac{q_\perp^2}{q^2 + g^2\Pi_\perp(q)} \qquad \vec{q} = (q_z, \vec{q}_\perp)$$

1. SF Bose-liquid:

$$\Pi_\perp(q\to 0) = n_s/m \quad \Rightarrow \quad P(q) \sim q^2 \Rightarrow \mathcal{D}_{zz} = \frac{4\pi T\cdot\mu_d}{q^2}$$

$$\mu_d = \left(1 + \frac{1}{4\pi\lambda^2 n_s}\right)^{-1}$$

2. "Normal" Bose-liquid:

$$\Pi_\perp(q\to 0) \simeq q^2 \quad \Rightarrow \quad P_\perp(q\to 0) \to const \Rightarrow$$

$$\underline{\mathcal{D}_{zz} = \frac{4\pi T}{q^2 + \lambda_{eff}^{-2}}}; \qquad \lambda_{eff}^2 = \lambda^2(1 + 4\pi\chi^B)$$

Superconductive current along $\vec{H}_{ext}$!

② Kane-Kivelson-Lee-Zhang (1991) Hamiltonian and SUSY

Suppose that $\Psi_0\{\vec{r}_i\} = \prod_{j>k} |\bar{r}_j - \bar{r}_k|^{2\alpha} e^{-\pi\alpha n \sum_i r_i^2}$

is an exact ground state wavefunction; what is the Hamiltonian then?

$$H = \frac{1}{2m} \sum_{j\alpha} q^+_{j\alpha} q_{j\alpha} \qquad \underline{q_{j\alpha}|0\rangle = 0 \;!}$$

$$q_{j\alpha} = -i\frac{\partial}{\partial r_{j\alpha}} + i\cdot 2\alpha\left(\sum_{k\neq j} \frac{\partial \ln|\bar{r}_j - \bar{r}_k|}{\partial r_{j\alpha}} - \pi n r_{j\alpha}\right)$$

$$q^+_{j\alpha} = -i\frac{\partial}{\partial r_{j\alpha}} - i\cdot 2\alpha\left(\quad -''- \quad \right)$$

SUSY!
$$\mathcal{H} = \frac{1}{2m}\{Q^+, Q\}$$

$$Q = \sum_{j\alpha} q_{j\alpha} a^+_{j\alpha} \qquad Q^+ = \sum_{j\alpha} q^+_{j\alpha} a_{j\alpha}$$

a, a^+ – Fermi operators

$$\mathcal{H}|0_F\rangle \equiv H = \frac{1}{2m}\sum_i \Big\{ -\nabla_i^2 + 2\alpha \sum_{j\neq i} \nabla_i^2 \ln|\bar{r}_i - \bar{r}_j| + {} + 4\alpha^2 \sum_{\substack{j\neq i\\ k\neq i}} \vec{\nabla}_i \ln|\bar{r}_i - \bar{r}_j| \cdot \vec{\nabla}_i \ln|\bar{r}_i - \bar{r}_k| + {} + (2\pi\alpha n)^2 r_i^2 \Big\}$$

Continuum representation:

$$U = \frac{1}{2}\int d^2r_1\, d^2r_2\, \rho_1 \rho_2\, V_2(\bar{r}_1 - \bar{r}_2) + {} + \int d^2r_1\, d^2r_2\, d^2r_3\, \rho_1\rho_2\rho_3\, V_3(\bar{r}_1 - \bar{r}_2, \bar{r}_1 - \bar{r}_3)$$

$$V_2(r) = \frac{4\pi\alpha}{m}\delta(r) + \frac{4\alpha^2}{m}\frac{1}{r^2} \qquad V_3(\bar{r}, \bar{r}') = \frac{2\alpha^2}{m}\frac{\bar{r}\cdot\bar{r}'}{r^2 r'^2}$$

$$\rho_{\bar{r}} = n + \delta\rho_{\bar{r}} \longrightarrow \boxed{V_2^{eff}(r) = V_2(r) - \frac{g^2}{2\pi}\ln r}$$

③ Matrix dynamics

$$\alpha = 1/2 \ : \quad |\Psi_0\{\vec{r}_i\}|^2 = \text{const} \cdot \prod_{j>k} |\bar{r}_j - \bar{r}_k|^2 e^{-\pi n \sum_i \bar{r}_i^2}$$

Coinsides witha the Ginibre distribution for eigenvalues of random complex matrices.

What about dynamics ?

Consider Gaussian ensemble of normal matrices: $[\hat{M}, \hat{M}^+] = 0$

Then $\hat{M} = \hat{U}^+ \hat{Z} \hat{U}$ ← unitary

$\hat{Z}$ ← diagonal

Quantum mechanics of normal matrices:

$$Z = \int \mathcal{D}M \mathcal{D}M^+ \exp(-S(M, M^+)) \prod_\tau \delta([M, M^+])$$

$$S(M, M^+) = \int \left[\frac{1}{4} \text{Tr}\left(\frac{\partial M^+}{\partial \tau} \cdot \frac{\partial M}{\partial \tau} \right) + \text{Tr}(M^+ M) \right] d\tau$$

$$\mathcal{D}M \mathcal{D}M^+ \rightarrow \mathcal{J}\{z_i\} \prod_i dz_i d\bar{z}_i \mathcal{D}\hat{U}$$

$$\mathcal{J}\{z_i\} = \text{const} \prod_{i>j} |z_i - z_j|^2 \doteq |\Delta\{z_i\}|^2$$

The Hamiltonian in "radial coordinates":

$$H = -\frac{2}{\sqrt{g}} \sum_i \left[\frac{\partial}{\partial z_i} \sqrt{g} \frac{\partial}{\partial \bar{z}_i} + \frac{\partial}{\partial \bar{z}_i} \sqrt{g} \frac{\partial}{\partial z_i} \right] + \sum_i |z_i|^2$$

$$\sqrt{g} = |\Delta\{z_i\}|^2$$

$$\rightarrow H = -\frac{4}{|\Delta\{z_i\}|} \sum_i \frac{\partial^2}{\partial z_i \partial \bar{z}_i} |\Delta\{z_i\}| + V\{z_i, \bar{z}_i\}$$

$$V\{z_i, \bar{z}_i\} = \sum_i \sum_{\substack{j \neq i \\ k \neq i}} \frac{1}{(z_i - z_k)(\bar{z}_i - \bar{z}_j)} + 4 \sum_{i \neq j} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \log|z_i - z_j| + \sum_i |z_i|^2$$

$$\Psi\{z_i, \bar{z}_i\} \equiv |\Delta\{z_i\}| \cdot \Phi\{z_i, \bar{z}_i\} \rightarrow \boxed{\text{KKLZ for } \alpha = 1/2}$$

④ Identity for correlation functions

Consider regular parts of the density-density and current-current correlation functions at equal times : (Taniguchi, Shastry, Altshuler)

$$K(\bar{r}-\bar{r}') = \langle 0 | \sum_{j\neq k} \delta(\bar{r}-\bar{r}_j)\delta(\bar{r}'-\bar{r}_k) | 0 \rangle = \underset{\updownarrow S(\vec{\rho})}{S(\bar{r}-\bar{r}')} - n\,\delta(\bar{r}-\bar{r}') + n^2$$

$$\Pi^{reg}_{\alpha\beta}(\bar{r}-\bar{r}') = \frac{(-i)^2}{4m^2}\langle 0 | \sum_{j\neq k}\left(\frac{\partial}{\partial r_{j\alpha}}\delta(\bar{r}-\bar{r}_j) + \delta(\bar{r}-\bar{r}_j)\frac{\partial}{\partial r_{j\alpha}}\right)\times$$

$$\left(\frac{\partial}{\partial r_{k\beta}}\delta(\bar{r}'-\bar{r}_k) + \delta(\bar{r}'-\bar{r}_k)\frac{\partial}{\partial r_{k\beta}}\right)|0\rangle =$$

$$= \Pi_{\alpha\beta}(\bar{r}-\bar{r}') - \frac{n}{m}E_{kin}\,\delta_{\alpha\beta}\,\delta(\bar{r}-\bar{r}')$$

$$E_{kin} = \frac{m}{2N}\sum_i \langle 0 | \bar{v}_i^2 | 0 \rangle$$

The identity:

$$\boxed{\Pi^{reg}_{\alpha\beta}(r) = \frac{\alpha}{m^2}\,\frac{\delta_{\alpha\beta} - 2\hat{r}_\alpha\hat{r}_\beta}{r^2}\,K(r)}$$

The proof (following Taniguchi-Shastry-Altshuler result for Calogero model):

1. $\mathcal{J}_\alpha(\vec{r}) \equiv \frac{1}{2m}\sum_i\left(q^+_{i,\alpha}\,\delta(\bar{r}-\bar{r}_i) + \delta(\bar{r}-\bar{r}_i)\,q_{i\alpha}\right)$

2. $q_{i,\alpha}|0\rangle = 0$

3. $[q_{i\alpha}, q^+_{j\beta}] = 4\alpha\left[(1-\delta_{ij})\mathcal{D}_{\alpha\beta}(\bar{r}_i-\bar{r}_j) - \delta_{ij}\cdot\sum_{e\neq i}\left(\mathcal{D}_{\alpha\beta}(\bar{r}_i-\bar{r}_e) - \pi n\,\delta_{\alpha\beta}\right)\right]$

$$\mathcal{D}_{\alpha\beta}(\vec{r}) = \nabla_\alpha\nabla_\beta \ln r = \frac{\delta_{\alpha\beta} - 2\hat{r}_\alpha\hat{r}_\beta}{r^2}$$

Consider the consequencies for correlations in the momentum space :

$$\Pi_{\alpha\beta}(\vec{q}) = \hat{q}_\alpha\hat{q}_\beta\,\Pi_{\parallel}(q) + (\delta_{\alpha\beta} - \hat{q}_\alpha\hat{q}_\beta)\,\Pi_{\perp}(q)$$

$$\Pi_{\alpha\beta}(\bar{q}) = \frac{2\pi\alpha}{m^2}\int\frac{d^2p}{(2\pi)^2}\frac{(\bar{p}+\bar{q})_\alpha(\bar{p}+\bar{q})_\beta}{(\bar{p}+\bar{q})^2}\left[S(p)-n\right] + \frac{2\pi\alpha}{m^2}n^2\frac{q_\alpha q_\beta}{q^2} +$$

$$+ \frac{n}{m}E_{kin}\,\delta_{\alpha\beta}$$

$$\Pi_\perp(q) = \frac{2\pi\alpha}{m^2}\int\frac{d^2p}{(2\pi)^2}\left[1-\frac{\left[(\bar{p}+\bar{q})\bar{q}\right]^2}{(\bar{p}+\bar{q})^2q^2}\right]\left[S(p)-n\right] + \frac{n}{m}E_{kin}$$

Limit $q\to 0$: $\Pi_\perp(q)\to 0$. Therefore

$$\underline{E_{kin}} = -\frac{\pi\alpha}{mn}\int\frac{d^2p}{(2\pi)^2}\left[S(p)-n\right] = -\frac{\pi\alpha}{mn}\left[K(r=0)-n^2\right] = \underline{\frac{\pi\alpha n}{m}} = \frac{\omega_c}{4}$$

General q:

$$\boxed{\Pi_\perp(q) = -\frac{\alpha}{2m^2}\int_0^q\left(S(p)-n\right)\left(1-\frac{p^2}{q^2}\right)p\,dp}$$

!

$$\Pi_\perp(q\to 0) = \frac{\alpha n}{8m^2}q^2$$

$$\Pi_\perp(q\to\infty) = \frac{\alpha\pi n^2}{2m^2}$$

$\boxed{\alpha = 1/2}$:

$$K(r) = n^2\left[1-e^{-\pi n r^2}\right]$$

$$S(p) = n\left[1-e^{-p^2/4\pi n}\right]$$

(Ginibre 65)

$$\boxed{\Pi_\perp(q) = \frac{\pi n^2}{2m^2}\left[1-\frac{1-e^{-q^2/4\pi n}}{q^2/4\pi n}\right]}$$

⑤ Long-wave length spectrum of exitations.

Consider longitudinal part of current's correlation function:

$$(*)\quad \Pi_{\parallel}(q) = \frac{2\pi\alpha}{m^2}n^2 - \Pi_{\perp}(q) = \frac{2\pi\alpha}{m^2}n^2 + \frac{\alpha}{2m^2}\int_0^q (S(p)-n)\left(1-\frac{p^2}{q^2}\right)p\,dp$$

Additional relations:

$$q^2\Pi_{\parallel}(q,\omega) = \omega^2 S(q,\omega)$$ – continuity

$$\int \frac{d\omega}{2\pi}\,\omega\, S(q,\omega) = \frac{nq^2}{2m}$$ – sum rule

$$(**)\quad \Pi_{\parallel}(q) = \int \frac{d\omega}{2\pi}\frac{\omega^2}{q^2} S(q,\omega) = \frac{n}{2m}\tilde{\omega}(q)$$

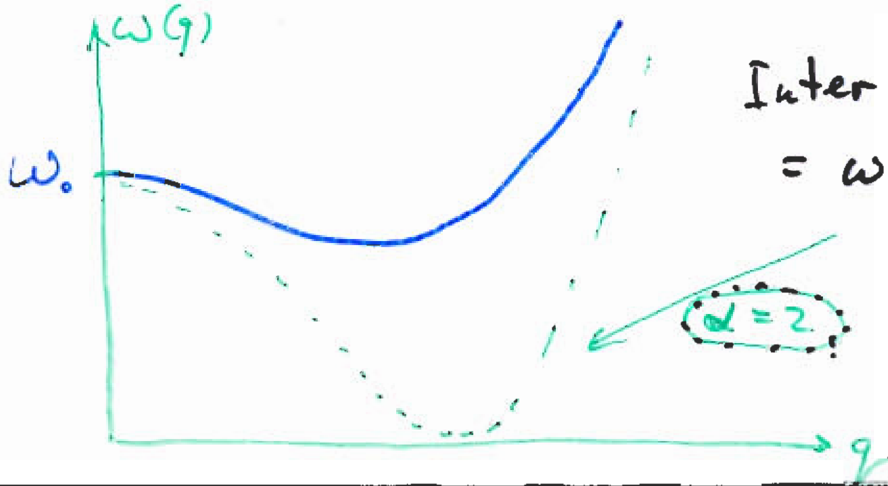
$$\boxed{\tilde{\omega}(q) \underset{\text{def.}}{\equiv} \frac{\int \omega^2 S(q,\omega)\,d\omega}{\int \omega S(q,\omega)\,d\omega}}$$

$$(*) + (**) \rightarrow \tilde{\omega}(q) = \underbrace{\frac{4\pi\alpha n}{m}}_{\omega_0} + \frac{\alpha}{mn}\int_0^q (S(p)-n)\left(1-\frac{p^2}{q^2}\right)p\,dp$$

Low q: $\boxed{\tilde{\omega}(q) = \omega_0 - \frac{\alpha q^2}{4m}}$

However at $q \to 0$ $\quad S(q,\omega) \Rightarrow 2\pi S(q)\,\delta(\omega-\omega(q))$

Therefore $\underline{\tilde{\omega}(q\to 0) = \omega(q\to 0)}$



Interpolation: $\omega^2(q) =$

$$= \omega_0^2 - \omega_0\frac{\alpha q^2}{2m} + \left(q^2/2m\right)^2$$

⑥ The limit of $\alpha \gg 1$: Quantum Hexatic State

What is the nature of the ground-state at $\alpha \to \infty$?

The structure factor $S(p)$ can be determined from

$$|\Psi_0\{\vec{r}_i\}|^2 \backsim \exp\Big\{4\alpha \sum_{j>k} \ln|\bar{r}_j - \bar{r}_k| - \pi\alpha n \sum_j r_j^2\Big\}$$

This is a partition function of a classical 2D gas with interaction $\boxed{V(r) = 4\alpha \log r}$ and $\boxed{T_{2D} = 1}$

At $4\alpha \gtrsim 140$ (Monte Carlo simulations)

2D Coulomb gas forms a crystal in the sense of Berezinsky - Kosterlitz - Thouless :

$$\boxed{S(\vec{p}) \backsim |\vec{p} - \vec{p}_a|^{-2+\eta}} \qquad (\eta \sim 1/\alpha)$$

It means that $\langle u^2 \rangle_{cl} \sim \log L$

Now consider the same $\langle u^2 \rangle$ calculated directly for the Quantum system :

$$\langle u^2 \rangle_{qu} \sim \int \frac{d^2q}{\omega(q)} \qquad \text{where} \quad \omega(q) \sim \underset{\uparrow}{\sqrt{\mu}} \cdot q$$

shear mode (transverse sound)

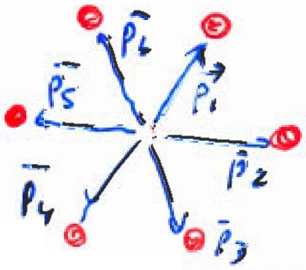
Thus $\langle u^2 \rangle \sim \log L$ implies $\mu(q) \sim q^2$, $\omega(q) \sim q^2 \Rightarrow$ zero shear modulus

$\vec{q} = \vec{p} - \vec{p}_a$

Direct calculation with

$$U\{\vec{r}_i\} = \frac{2\alpha^2}{m} \sum_i \Big[\sum_{j \neq i} \frac{\bar{r}_j - \bar{r}_i}{(\bar{r}_j - \bar{r}_i)^2}\Big]^2 + \frac{2\pi\alpha}{m} \sum_{j \neq i} \delta(r_i - r_j)$$

produces the same result.



(7) Relation to the stochastic dynamics

Consider a set of Langevin Eqs.:

$$\frac{dr_{j\alpha}}{dt} = -\frac{\partial W}{\partial r_{j\alpha}} + \xi_{j\alpha}(t)$$

$$\overline{\xi_{j\alpha}\xi_{k\beta}} = 2T\delta_{jk}\delta_{\alpha\beta}\delta(t-t')$$

The equivalent Fokker-Planck Eq.:

(*) $$\frac{\partial \mathcal{P}}{\partial t} = \sum_{j\alpha}\frac{\partial}{\partial r_{j\alpha}}\left(\frac{\partial W}{\partial r_{j\alpha}} + T\frac{\partial}{\partial r_{j\alpha}}\right)\mathcal{P}\{\vec{r}_i\}$$

After substitution $\mathcal{P} = e^{-W/2T}\cdot\Psi$

(*) reduces to imaginary-time Schrödinger Eq. with potential

$$U = \frac{1}{4}\sum_{j\alpha}\left(\frac{\partial W}{\partial r_{j\alpha}}\right)^2 - \frac{T}{2}\sum_{j\alpha}\frac{\partial^2 W}{\partial r_{j\alpha}^2}$$

(Feigelman-Tsvelick, 1982)

Choose $W = -4\alpha\sum_{j>k}\ln|\vec{r}_j-\vec{r}_k| + 2\pi\alpha n\sum_j r_j^2$

then we get $U\{\vec{r}_i\}$ of our Hamiltonian !

The outcome: $\langle\Psi_0|\mathcal{F}\{\vec{r}_i\}|\Psi_0\rangle = \langle\mathcal{F}\{\vec{r}_i\}\rangle_{Lang.}$

Quantum expectation values can be computed from simulations of classical Langevin Eqs.

Two simple examples:

1. The relation $E_{kin} = \omega_0/4$ was checked numerically for $\alpha \in (0.1, 100)$, using $E_0 = E_{kin}+E_{pot} = 0$
2. Static structure factor $S(p)$ was computed from numerical simulations, and then used to calculate $\Pi_\perp(p)$

⑧ Speculations and Conclusions

1. Low-q spectrum is given exactly by

$$\omega(q) = \omega_0 - \alpha\frac{q^2}{4m}, \quad q \to 0$$

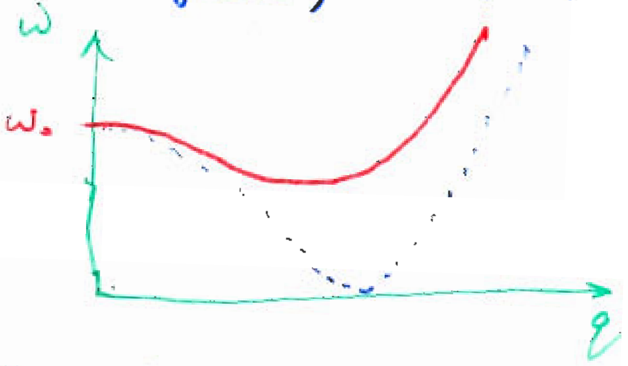
The perturbative calculation for purely log r interaction (Skvortsov and Feig., unpublished) gives very similar result in the 1st order in α!

$$\omega(q) - \omega_0 = 0.81... \cdot \alpha \cdot q^2/4m, \quad q \to 0.$$

The result for the present model suggests that SUSY structure of the model leads to cancellation of all higher-order diagramms, at least for the spectrum $\omega(q)$.

Explicitely Supersymmetric diagramm technique is needed to study this point.

2. At small α the ground-state is a SF liquid, with a gap in the spectrum:



At $\alpha \gtrsim 2 \div 3$ an appearence of a gapless mode is expected

3. At $\alpha > \alpha_{cr} \simeq 35$ the non-SF Quantum Hexatic state exists; there are anisotropic soft modes: $\omega(q) \sim (\vec{q} - \vec{p}_a)^2$

What is the nature of the ground-state at $\alpha \in [3, 35]$??

An existence of a isotropic non.-SF liquid with a gapless mode is possible.

A very interesting possibility: spontaneous breakdown of the Galilei invariance, without any symmetry breaking in real space.

$$\frac{n_s}{m} = \frac{n}{m} - \mathcal{P}(q\to 0, \omega=0)$$

$$\mathcal{P}(q,\omega=0) = \frac{2}{\pi}\int_0^\infty \frac{\Pi_\perp(q,\omega)}{\omega}\,d\omega$$

But $\Pi_\perp(q\to 0) = \frac{1}{\pi}\int_0^\infty \Pi_\perp(q\to 0,\omega)\,d\omega \to 0$!

How to get non-zero $\mathcal{P}(q\to 0,\omega=0)$, i.e. $n_s<n$?

The function $\Pi_\perp(q,\omega)$ should be singular function of q^2/ω, e.g.:

$$\Pi_\perp(q,\omega) = \frac{q^2/M\omega}{1+(q^2/M\omega)^2} + \tilde{\Pi}_\perp(q,\omega)$$

Such a behaviour implies an existence of a gapless mode $\Rightarrow$ Galilei inv. breakdown.

4. At $\alpha=1/2$ there is exact mapping to the quantum mechanics of normal matrices.

Two possible routes of development:

a) Dynamics of normal matrices migth be exactly solvable.

b). Similar mapping may exist for other values of α (like in the case of Calogero-Sutherland model).