Conformal invariance in two-dimensional turbulence

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Conformal invariance in statistical physics

- Homogeneity + isotropy + scale invariance $\rightarrow$ conformal invariance.Holds under broad conditions (Polyakov, Polchinsky) yet there are counterexamples (2D elasticity, Riva and Cardy)

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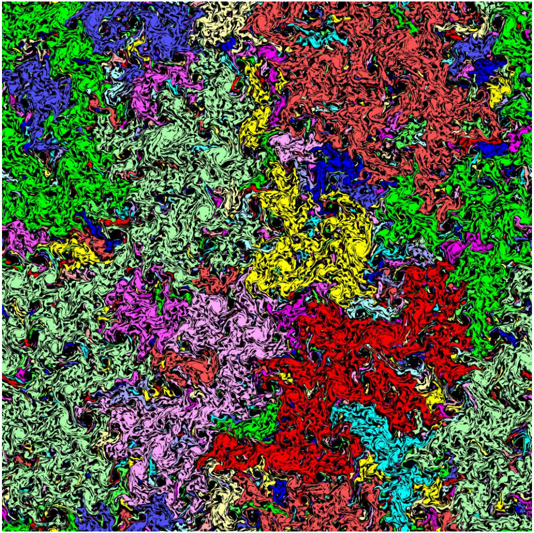
- Here: Inverse cascade
- Here: Stochastic Löwner Evolution: domain interfaces as conformally invariant curves (Schramm)

Vorticity clusters in two-dimensional turbulence



Clusters

- $\partial_t \omega + \boldsymbol{v} \cdot \boldsymbol{\nabla} \omega = \nu \nabla^2 \omega + f$
- f acts at scale L_f
- Scaling limit $L_f \to 0$
- $\omega_r \sim r^{-2/3}$
- Boundaries are isovorticity lines
- Fjords, lakes, necks

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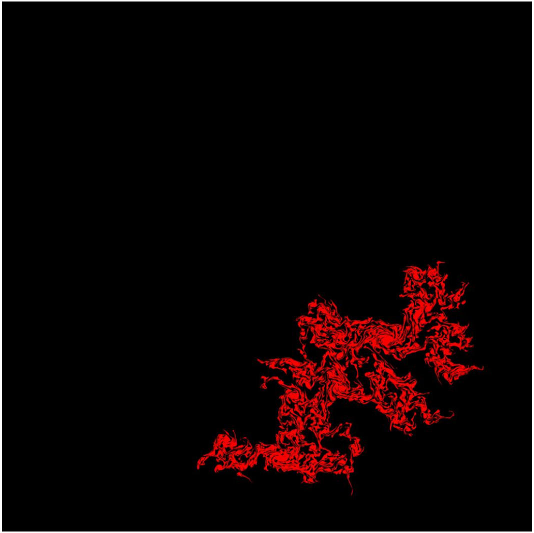

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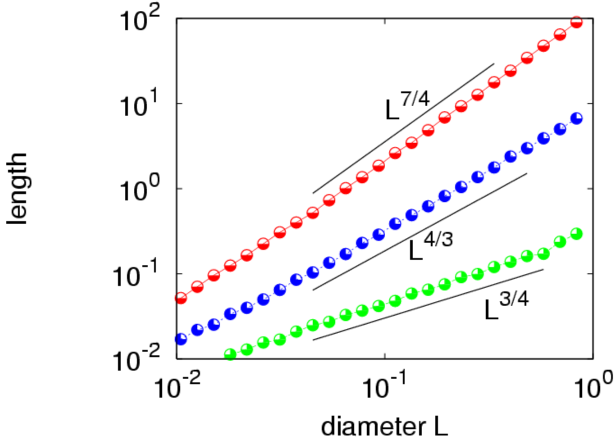
Fractal dimensions of vorticity clusters



Dimensions

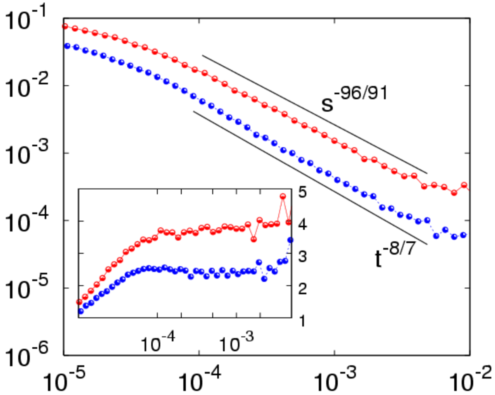
- Boundary
- Frontier
- Cut points

Brownian motion

Dimensions

- Pioneer points: $7/4$
- Frontier points: $4/3$
- Cut-points: $3/4$
- Proven by Lawler Schramm Werner via SLE
- Frontier = SAW (proves Mandelbrot's conjecture)
- Same exponents as percolation [Nienhuis (Coulomb gas), Duplantier (2D QG), Smirnov (SLE)]
- Conformally invariant

Surface and boundary of vorticity clusters



Probability distributions

- Power-law pdfs
- Exponents close to the exact values known for critical percolation

Vorticity clusters and independent percolation

- Independent percolation: short-correlated gaussian fields

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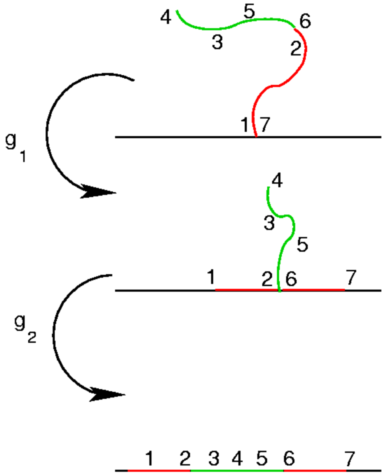
- Same class of universality as independent percolation or just close ?
- Need of more precise measurements
- Conformal invariance of zero-vorticity lines?
  Stochastic Löwner Evolution

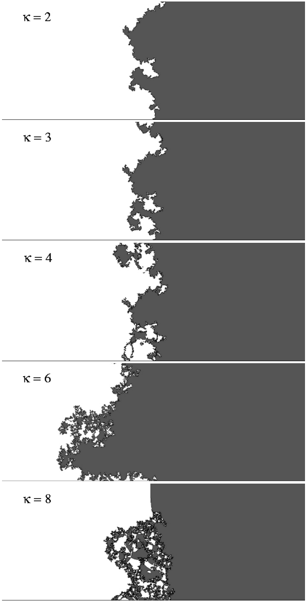
Löwner evolution



Growing hulls

- Riemann mapping theorem
  $g_t : \mathbb{H} \backslash K_t \to \mathbb{H}$
- normalization and reparameterization
  $g_t(z) \overset{z\to\infty}{\sim} z + 2t/z + O(1/z^2)$
- Löwner equation: $\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}$
- Trace and driving: $g_t(\gamma(t)) = \xi(t)$

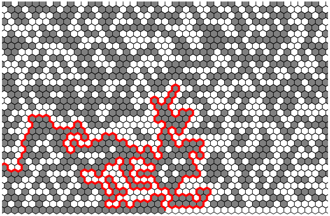
Stochastic Löwner Evolution (SLE$_\kappa$)



Chordal SLE in the half-plane

- $\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi(t)}$
- Composition property implies $\xi(t)$ has identical independent increments, together with reflection invariance and continuity ...
- $\xi(t) = \sqrt{\kappa} B_t$, B_t standard Brownian motion (Schramm)
- $0 < \kappa < 4$ simple curves
- $4 < \kappa < 8$ nonsimple curves
- $\kappa > 8$ space-filling
- Fractal dimension $D_F = \min[1 + \frac{\kappa}{8}, 2]$
- Duality $\kappa_\star = \frac{16}{\kappa}$

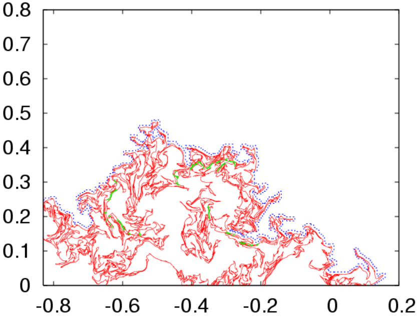
SLE$_\kappa$ and 2D critical systems



Interfaces

- $\kappa = 2$ Loop-erased random walk $\star$
- $\kappa = \frac{8}{3}$ Self-avoiding random walk $\star$
- $\kappa = 3$ Cluster boundaries in the Ising model
- $\kappa = 4$ Roughening (SOS), Gaussian free field isolines
- $\kappa = 6$ Cluster boundaries in independent percolation $\star$
- $\kappa = 8$ Uniform spanning trees $\star$
- CFT $\Leftrightarrow$ SLE (Bauer Bernard): $c = \frac{(\kappa-6)(8-3\kappa)}{2\kappa}$

From interfaces to driving functions



How to determine κ

- Approximate the interface as a polygonal
- Composition of conformal maps
  $g_t = g_{h_n} \circ g_{h_{n-1}} \circ \cdots \circ g_{h_1}$,
  $t = h_1 + h_2 + \ldots + h_n$
- Find the driving $\xi(t) = g_t(\gamma(t))$

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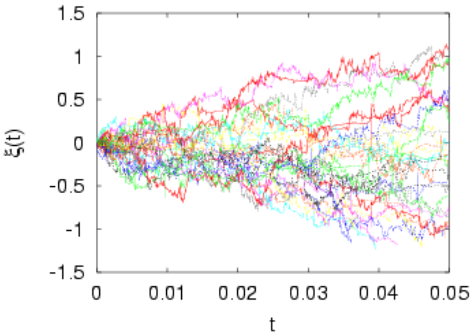

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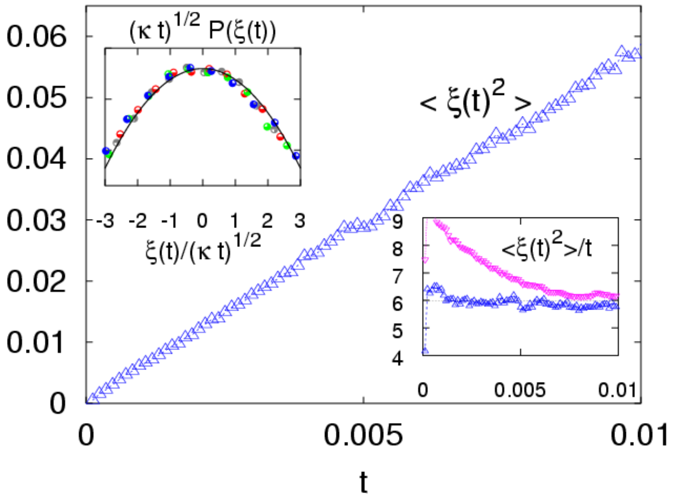
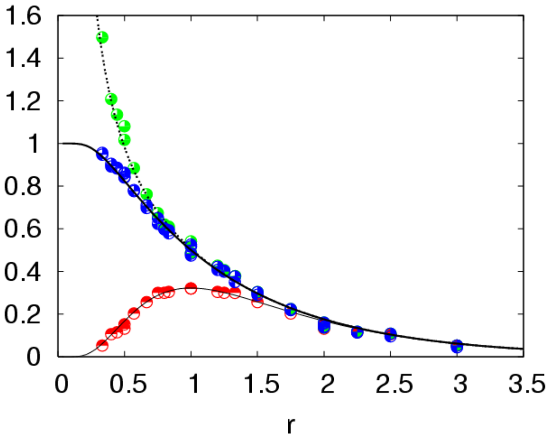
Zero-vorticity lines are SLE$_6$



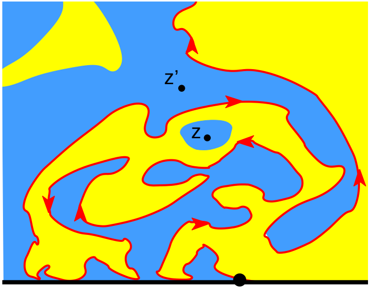
Crossing formulae

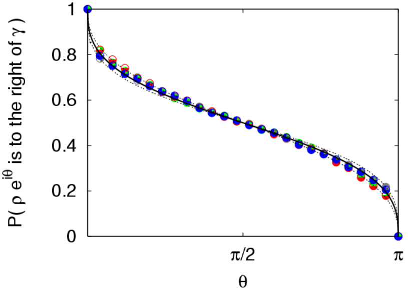
Spanning clusters

- Rectangle: r=height/width
- Vertical crossing
- Four-legged cluster
- Number of vertically spanning clusters
- Cardy, Watts (CFT) – Smirnov, Dubédat (SLE)



Surrounding cluster

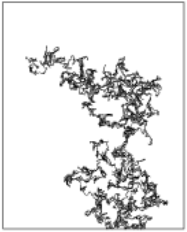


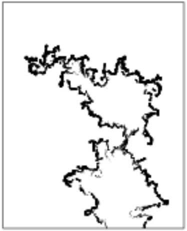


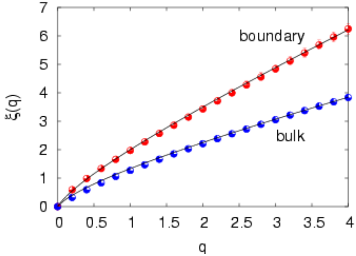
Right passage of SLE

Probability that $z = \rho e^{i\theta}$ is surrounded by a yellow cluster connected to the positive real axis $\Leftrightarrow$ Probability that z is on the right of the interface [Schramm (SLE)]

Harmonic measure on vorticity clusters







Multifractality

- Exponents for $\langle \mu_r^q \rangle$: Duplantier (QG), Lawler Schramm Werner (SLE), Wiegmann *et al* (CFT)
- Im(g) potential, $|g'|$ modulus electric field, $\mu_r \sim |g'|r$

Conclusions

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- General method for extracting driving functions from putative SLE traces

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Open issues

- Beyond interfaces: are correlation functions conformally invariant as well ?
- CFT at $c = 0$ has an operator with the same scaling as vorticity in the inverse cascade ($\phi_{1;3}$). Is there ground for a CFT theory of 2d turbulence ?