Two-dimensional turbulence: a review

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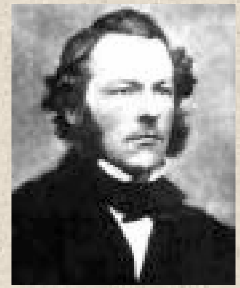
3d equations of motion

The equation of motion for an incompressible 3d flow are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

($\nabla \cdot \mathbf{u} = 0$, ν viscosity)



Claude Navier, 1827

George G. Stokes, 1845

3d NS equations conserves kinetic energy $E = \frac{1}{2} \int |\mathbf{u}|^2 \, d^3x$ in the inviscid limit.

The viscous energy balance is

$$\frac{dE}{dt} = -2\nu Z$$

enstrophy $Z = \frac{1}{2} \int |\nabla \times \mathbf{u}|^2 \, d^3x$

Navier-Stokes equations and turbulence

The flow becomes turbulent when

$$\mathrm{Re} = \frac{\left[\mathbf{u}\cdot\nabla\mathbf{u}\right]}{\left[\nu\nabla^2\mathbf{u}\right]} ; \ \frac{UL}{\nu} \ ? \ 1$$

In this limit (i.e. ν->0) the energy dissipation rate remains finite:

$$\varepsilon = \lim_{\nu\to 0} 2\nu Z > 0$$

dissipative anomaly

Typical Re numbers
laboratory: Re ≈ 10^4
atmosphere: Re ≈ 10^7

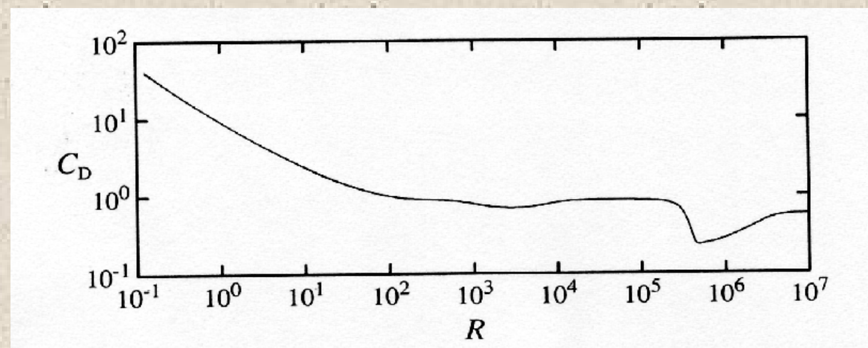
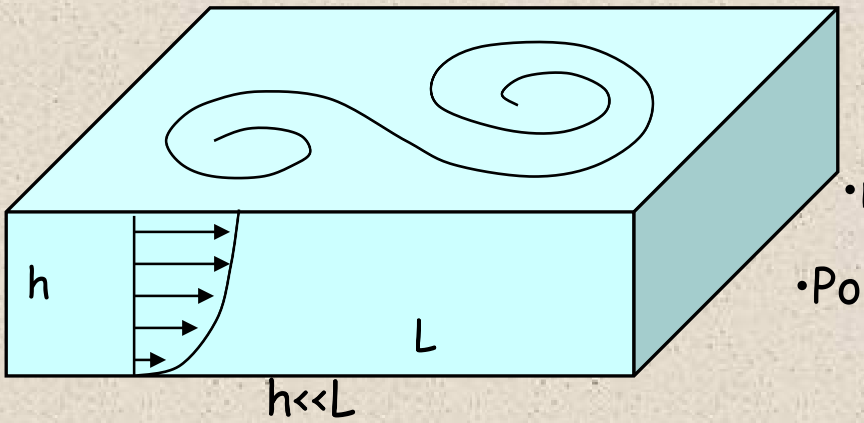


Fig. 5.11. Variation of drag coefficient with Reynolds number for circular cylinders.

Ekman-Navier-Stokes equations

Shallow layer of incompressible fluid



3D Navier Stokes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\nabla\cdot\mathbf{u} = 0$$

•neglect vertical motions $u_z = O(h/L)u_{x,y} : 0$

•Poiseuille's velocity profile $u_{x,y}(z) : z^2$

$$\nu\nabla_{(3d)}^2\mathbf{u} -> \nu\nabla_{(2d)}^2\mathbf{u} - \alpha\mathbf{u}$$

$$\boxed{\alpha \sim \nu/h^2}$$

$\boxed{\text{2D Navier Stokes + Friction}}$

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\dfrac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u} - \alpha\mathbf{u} + \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

α

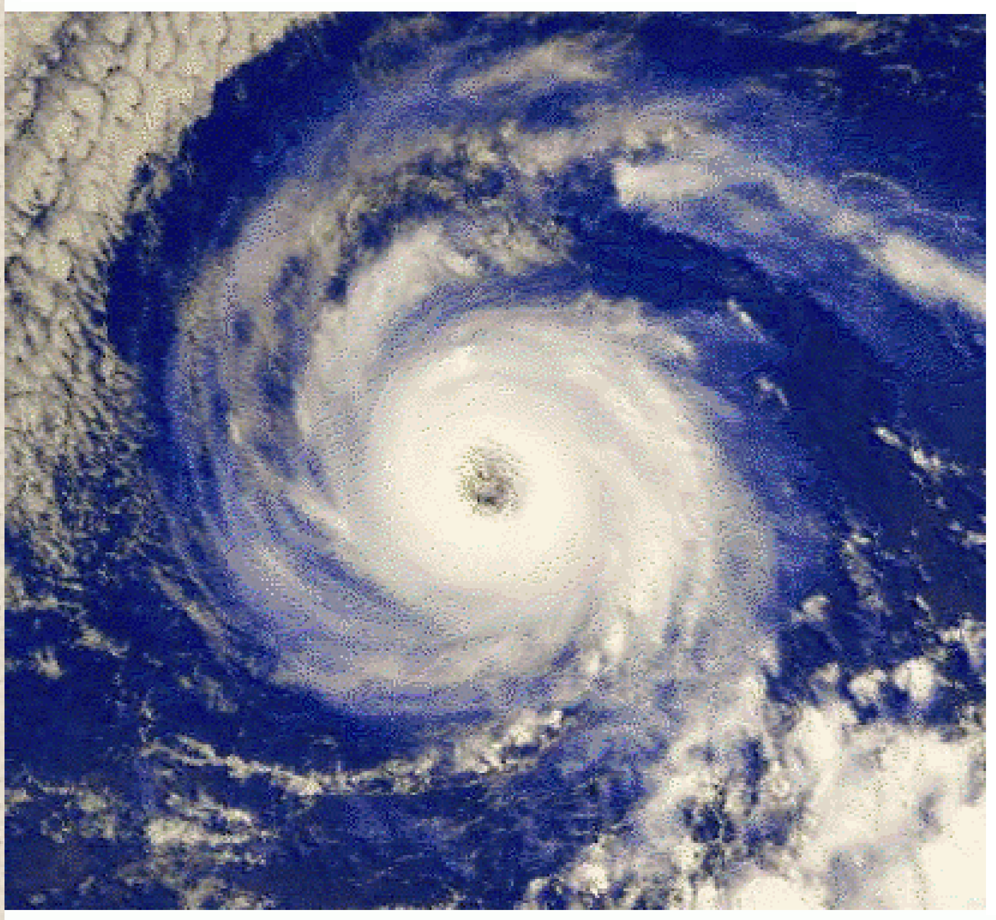
Ekman friction (rotating flow)
Rayleigh friction (stratified flow)
Hartmann friction (MHD)
air friction (soap film)

For vorticity $\omega = \nabla\times\mathbf{u}$

$$\boxed{\frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla\omega = \nu\nabla^2\omega - \alpha\omega + f}$$

Two-dimensional turbulence and geophysical flows

2D Navier-Stokes equation are a simple model for large scale motion of atmosphere and oceans: thin layers of fluid in which stratification and rotation supress vertical motions.



Energy/enstrophy balance for 2d NS

2d Navier-Stokes equations have two inviscid quadratic invariants:

Energy $$E = \frac{1}{2}\int |\mathbf{u}|^2 d^2x = \int E(k)dk$$ E(k): energy spectrum

Enstrophy $$Z = \frac{1}{2}\int \omega^2 d^2x = \int k^2 E(k)dk$$

Energy/enstrophy balance in viscous flow:

$$\frac{dE}{dt} = -2\nu Z \qquad \frac{dZ}{dt} = -2\nu P$$

$$P = \frac{1}{2}\int |\nabla \times \omega|^2 d^2x \geq 0$$

palinstrophy

In fully developed turbulence limit, Re=UL/ν -> ∞ (i.e. ν->0):

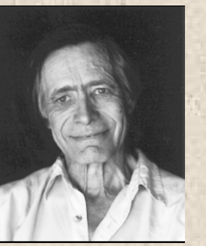
$$\boxed{\lim_{\nu \to 0} \frac{dE}{dt} = 0}$$

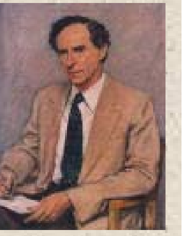
(because dZ/dt≤0 and Z(t) ≤Z(0))

no dissipative anomaly for energy in 2d: no energy cascade to small scales !

The double cascade

(Kraichnan 1967, Leith 1968, Batchelor 1969)

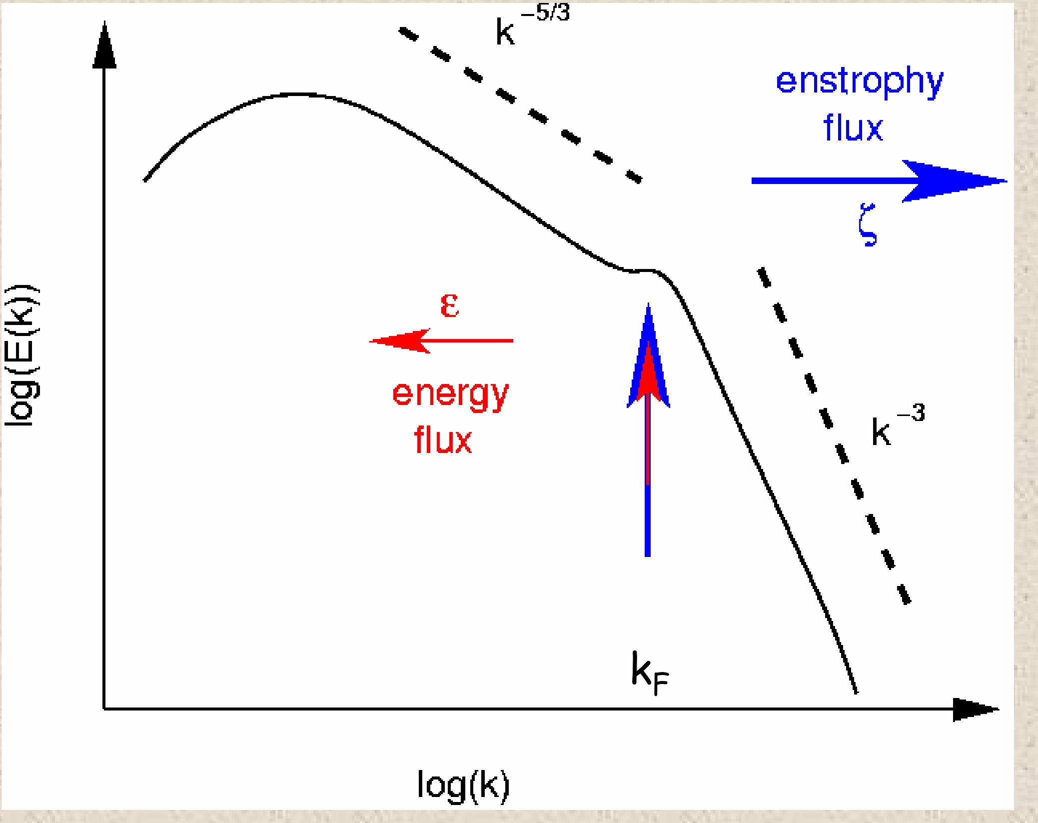




In the limit of Re->∞ 2d turbulence displays an enstrophy cascade to small scales at a rate ζ.
Energy flows to large scales with rate ε generating the inverse cascade.

Two inertial range of scales:

- energy inertial range $1/L<k<k_F$ (with constant ε)
- enstrophy inertial range $k_F<k<k_d$ (with constant ζ)



Two power-law self similar spectra in the inertial ranges.

The double cascade scenario is typical of 2d flows, e.g. plasmas and geophysical flows.

Exact results

Following the derivation obtained by Kolmogorov for 3d turbulence (Kolmogorov 4/5 law) is it possible to obtain for 2d cascades two exact results:

inverse energy cascade:

$$S_3(r) = \left\langle \left(\delta_r u\right)^3 \right\rangle = \frac{3}{2}\varepsilon r$$

direct enstrophy cascade:

$$\left\langle \delta_r u \left(\delta_r \omega\right)^2 \right\rangle = -2\zeta r$$

Kolmogorov's 4/5 law (1941)

Dissipation of energy in the locally isotropic turbulence†

By A. N. Kolmogorov

In my note (Kolmogorov 1941*a*) I defined the notion of local isotropy and introduced the quantities

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[u_d(M')-u_d(M)]^2}, \\ B_{nn}(r) &= \overline{[u_n(M')-u_n(M)]^2}, \end{aligned}\right\} \qquad (1)$$

where r denotes the distance between the points M and M', $u_d(M)$ and $u_d(M')$ are the velocity components in the direction $\overline{MM'}$ at the points M and M', and $u_n(M)$ and $u_n(M')$ are the velocity components at the points M and M' in some direction, perpendicular to MM'.

In the sequel we shall need the third moments

$$B_{ddd}(r) = \overline{[u_d(M')-u_d(M)]^3}. \qquad (2)$$

For the locally isotropic turbulence in incompressible fluid we have the equation

$$4\bar{E} + \left(\frac{\mathrm{d}B_{ddd}}{\mathrm{d}r} + \frac{4}{r}B_{ddd}\right) = 6\nu\left(\frac{\mathrm{d}^2B_{dd}}{\mathrm{d}r^2} + \frac{4}{r}\frac{\mathrm{d}B_{dd}}{\mathrm{d}r}\right) \qquad (3)$$

similar to the known equation of Kármán for the isotropic turbulence in the sense of Taylor. Herein $\bar{E}$ denotes the mean dissipation of energy in the unit of time per unit of mass. The equation (3) may be rewritten in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{4}{r}\right)\left(6\nu\frac{\mathrm{d}B_{dd}}{\mathrm{d}r} - B_{ddd}\right) = 4\bar{E}, \qquad (4)$$

and, in virtue of the condition $(\mathrm{d}/\mathrm{d}r)B_{dd}(0) = B_{ddd}(0) = 0$, yields

$$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r - B_{ddd} = \tfrac{4}{5}\bar{E}r. \qquad (5)$$

For small r we have, as is known,

$$B_{dd} \sim \tfrac{1}{15}\bar{E}r^2/\nu, \qquad (6)$$

i.e. $$6\nu\,\mathrm{d}B_{dd}/\mathrm{d}r \sim \tfrac{4}{5}\bar{E}r.$$

Thus, the second term on the left-hand side of (5) is for small r infinitesimal in comparison with the first. For large r, on the contrary, the first term may be neglected in comparison with the second, i.e. we may assume that

$$B_{ddd} \sim -\tfrac{4}{5}\bar{E}r. \qquad (7)$$

It is natural to assume that for large r the ratio

$$S = B_{ddd} : B_{dd}^{3/2}, \qquad (8)$$

† First published in Russian in *Dokl. Akad. Nauk SSSR* (1941), **32**(1). Paper received 30 April 1941. This translation by V. Levin, reprinted here with emendations by the editors of this volume.

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Printed in Great Britain

Dimensional derivation

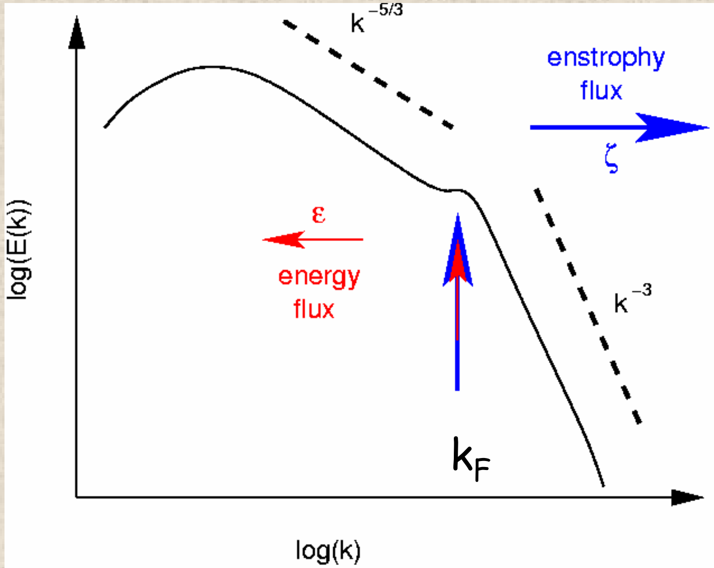
enstrophy cascade inertial range:
constant enstrophy flux

$$\zeta \,;\, \frac{(\delta_r \omega)^2}{\tau_r} \,;\, \frac{\delta_r u (\delta_r \omega)^2}{r}$$

η« r « r_F

dimensionally: $\delta_r u/r=\delta_r\omega$ and thus
(scaling exponent h=1)

$$\delta_r \omega \,;\, \zeta^{1/3}$$
$$\delta_r u \,;\, \zeta^{1/3} r$$



energy spectrum $\boxed{E(k) \,;\, \zeta^{2/3} k^{-3}}$

Kraichnan 1971
Logarithmic corrections

$$E(k) \,;\, \zeta^{1/3} k^{-3} \ln^{-1/3}(kL)$$

energy cascade inertial range:
constant enstrophy flux

$$\varepsilon \,;\, \frac{(\delta_r u)^2}{\tau_r} \,;\, \frac{(\delta_r u)^3}{r}$$

r_F « r « L

thus $\delta_r u \,;\, \varepsilon^{1/3} r^{1/3}$ (scaling exponent h=1/3)

energy spectrum $\boxed{E(k) \,;\, \varepsilon^{2/3} k^{-5/3}}$

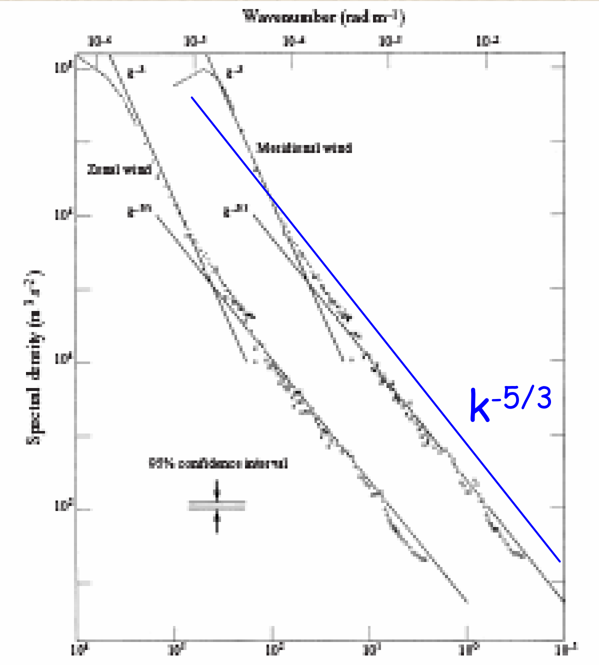
Experiments

Atmospheric flows:

QuickTime™ and a
YUV420 codec decompressor
are needed to see this picture.

Mesoscale wind variability
(radar and balloon): $k^{-5/3}$
K.S. Gage, J.Atmos.Sciences **36** (1979)

GASP aircraft dataset: $k^{-5/3}$ for
wavelenghts 3-300 km
Nastrom, Gage, Jasperson, Nature 310 (1984)



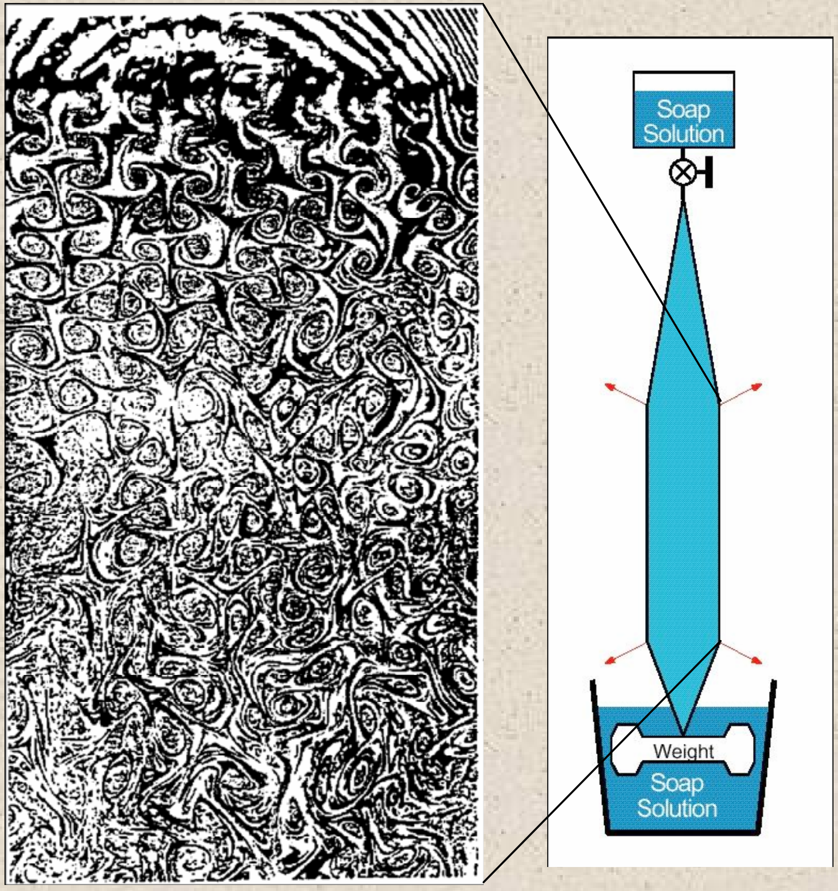
Lenght (km)

Laboratory experiments

Soap films

(Y. Couder, W. Goldburg, H. Kellay,
M.A. Rutgers, M. Rivera, R.E. Ecke)

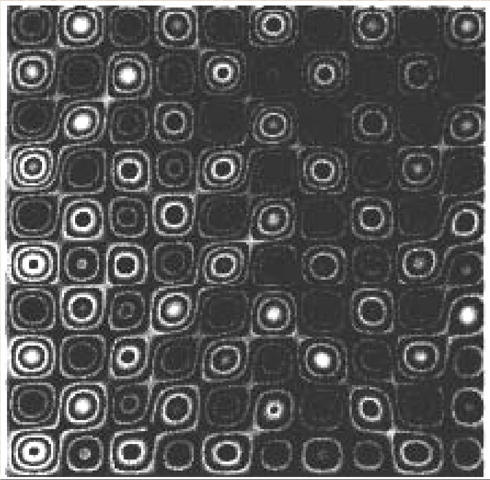
interferometry, LDV, PIV

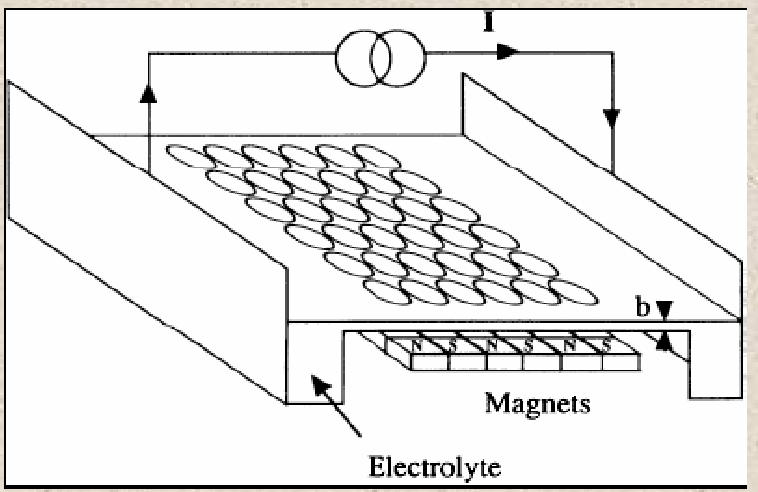


Thin layer of electrolyte
driven by Lorentz force

(P. Tabeling, J. Gollub,
A. Cenedese)

PIV





Direct numerical simulations

Integration of 2d-NS equations for vorticity scalar field $\omega=-\nabla^2\psi$

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega + f$$

stream function ψ with $\mathbf{u}=(\partial_y\psi, -\partial_x\psi)$

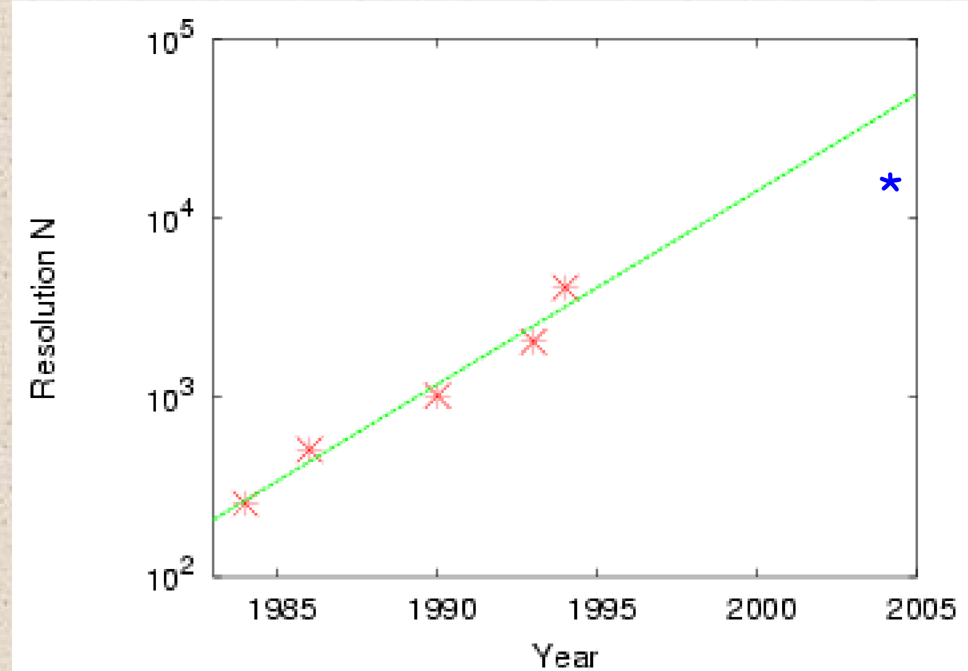
Homogeneous, isotropic turbulence: square box with periodic boundary conditions

Pseudo-spectral code

* Fourier decomposition: derivatives in Fourier spaces
* Convolutions (products): back in physical space with FT

Extensive use of Fast Fourier Transform
Computationally much faster than 3d simulations

A brief history of DNS



DNS of 2d turbulence followed and explosive trend in the early stage.

From Frisch & Sulem (Phys. Fluids 27 1984) at 256^2 to Borue (Phys. Rev. Lett. 71, 1993) at 4096^2 the resolution grows following an exponential law:

$$N = 267 * 2^{(t-1984)/2.8}$$

Late stage: saturation at “reasonable” resolution (present simulations at 2048^2 on PC)

High resolution DNS

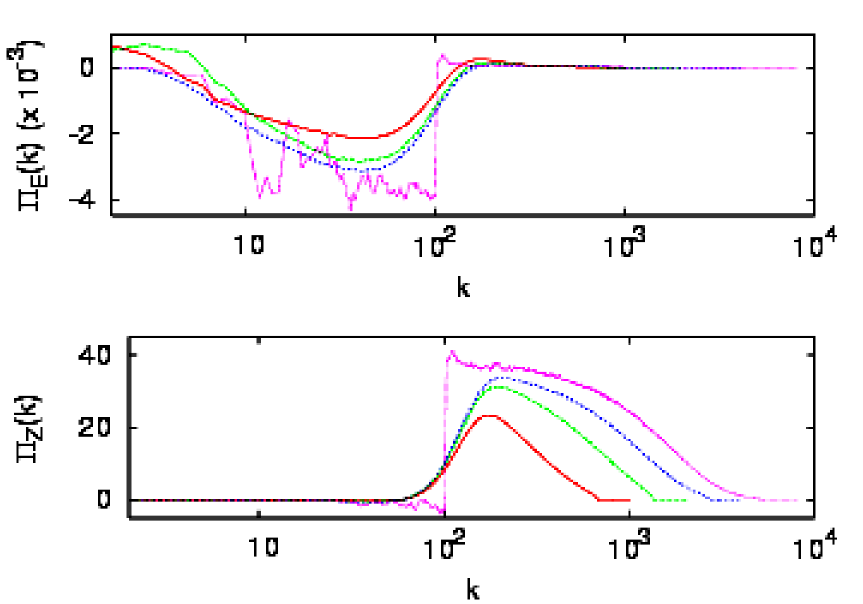
(G. Boffetta and A. Celani, 2005)

Set of simulations with resolutions up to 16384^2 with a parallel pseudo-spectral code on IBM-SP4 at Cineca.

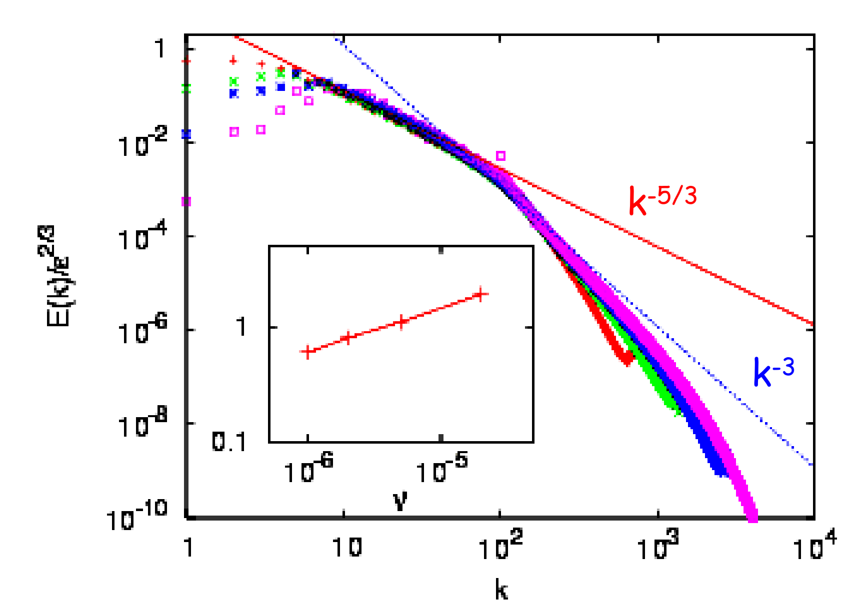
N	ν	α	r_F	η	ε	ζ
2048	2×10^{-5}	0.015	0.01	2.4×10^{-3}	2.1×10^{-3}	38.0
4096	5×10^{-6}	0.024	0.01	1.2×10^{-3}	3.2×10^{-3}	36.1
8192	2×10^{-6}	0.025	0.01	7.8×10^{-4}	3.6×10^{-3}	35.3
16384	1×10^{-6}	0.0	0.01	5.5×10^{-4}	3.6×10^{-3}	37.6

Simultaneously observation of direct and inverse cascade

Energy/enstrophy fluxes in spectral space



Energy spectra



[redacted]

Direct cascade

Early numerical simulations failed to reproduce k^{-3} spectrum

Steeper spectrum (k^{-4}-k^{-5}) associate with the presence of strong large scale vortices

Decaying turbulence shows long term memory of initial conditions (no universality)

(Benzi, McWilliams)

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Recent simulations more in agreement with Kraichnan prediction (logarithmic corrections?)

S.Chen, R. Ecke, G. Eyink, X. Wang, Z. Xiao, PRL **91**, (2003)
E.Lindborg, K.Alvelius, Phys. Fluids **12** 945 (2000)
C.Pasquero, G.Falkovich, PRE **65** 056305 (2002)

Direct cascade with linear friction:

when friction is sufficiently strong small scale vorticity has the same statistics of a passive scalar with finite lifetime (linear problem)

2d vorticity equation: analogy with passive scalar

S. Corrsin, JFM **11**, 407 (1961) - M. Chertkov, Phys. Fluids **10**, 3017 (1998)
D. Bernard, Europhys. Lett. **50** 333 (2000)
K. Nam, E. Ott, T.M. Antonsen, P.N. Guzdar, PRL **84**, 5134 (2000)
G.Boffetta, A.Celani, S.Musacchio, M.Vergassola, Phys. Rev. E **66**, 026304 (2002)

Vorticity Navier-Stokes with friction

ACTIVE

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Sorenson Video 3 decompressor
are needed to see this picture.

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \nabla^2 \omega - \alpha \omega + f_\omega$$

- Non linear equation for $\omega = \nabla \times \mathbf{u}$
- friction α

Passive scalar with finite lifetime

PASSIVE

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta - \alpha \theta + f_\theta$$

- Linear equation for θ
- lifetime $\tau \sim 1/\alpha$

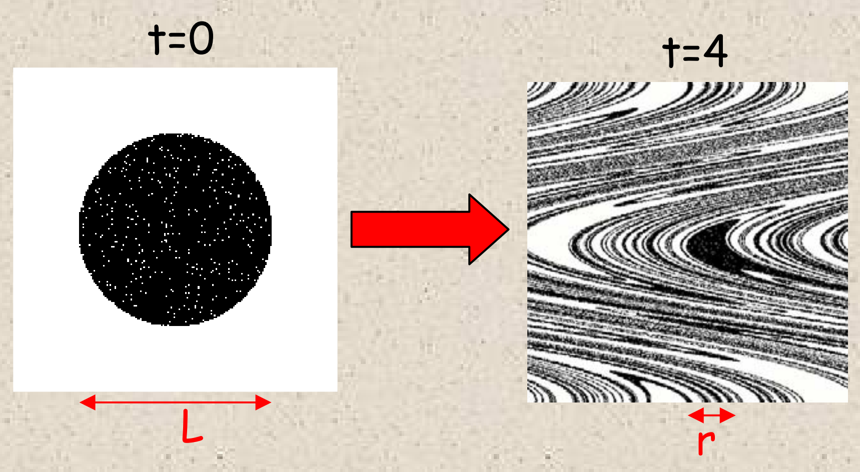
e.g. biological (plankton), chemical pollutants

Mean field computation for passive scalar with finite lifetime

S. Corrsin, JFM **11**, 407 (1961)

Stretching: incompressible smooth velocity generates small scale features at exponential rate (Lyapunov):

$$r = L\, e^{-\lambda t}$$



Linear damping: fluctuations decay at exponential rate

$$\theta(\mathbf{x}(t),t) = \theta(\mathbf{x}(0),0)\, e^{-\alpha t}$$

Combining the two exponential behavior, one has $\delta\theta(r,t) = \delta\theta(L,0)\left(\frac{r}{L}\right)^{\alpha/\lambda}$

and thus, in stationary conditions:

$$\left\langle \left(\delta\theta(r,t)\right)^{2p} \right\rangle \approx r^{2p\alpha/\lambda}$$

α-dependent structure function exponents (and spectral index)

Fluctuations of Lyapunov exponent: intermittency corrections

M. Chertkov, Phys. Fluids **10**, 3017 (1998)

Lagrangian description of vorticity cascade

Statistics of vorticity fluctuations $\delta\omega(r)$
at small scale $r < L_F$ (forcing correlation lenght)

Vorticity fluctuations $\delta\omega(r)$ come from forcing contributions along two fluid trajectories (ν=0):

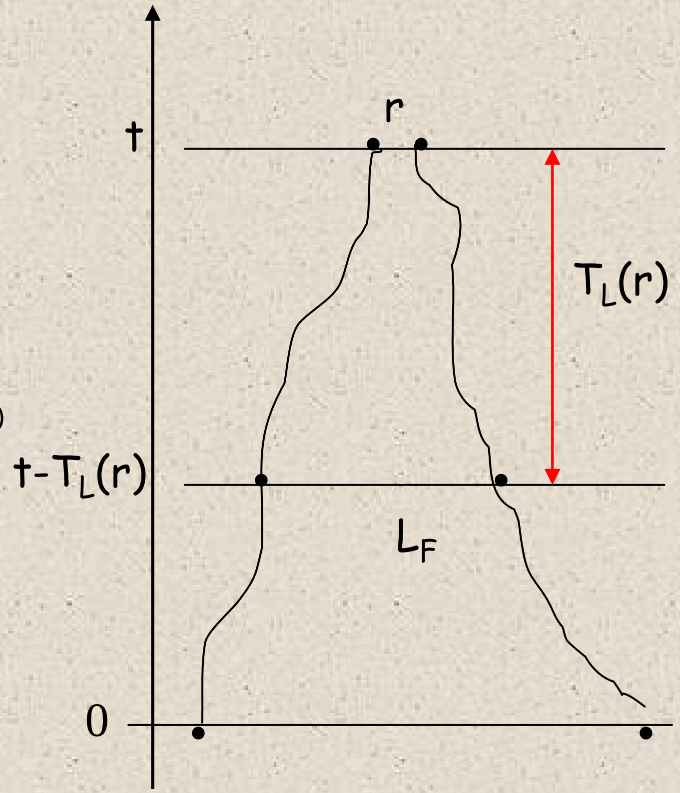
$$\delta\omega(r) = \omega(\mathbf{x}', t) - \omega(\mathbf{x}, t) = \int_0^t ds \left[f(\mathbf{x}'(s), s) - f(\mathbf{x}(s), s) \right] e^{-\alpha(t-s)}$$

Forcing gives different contributions only at separations > L_F:

$$\delta\omega(r) \approx \Omega \int_0^{t-T_L(r)} ds\, e^{-\alpha(t-s)} = \Omega e^{-\alpha T_L(r)}$$

$$\boxed{\delta\omega(r) \approx \Omega e^{-\alpha T_L(r)}}$$

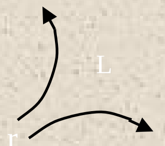
- $\langle\Omega^2\rangle$ = forcing variance
- $T_L(r)$ = Exit-time



$T_L(r)$ is the time that a couple of particles at distance r takes to reach a separation L_F backward in time.

In a two-dimensional incompressible smooth flow particles separate exponentially fast

$$L = re^{\gamma t}$$



$$\boxed{T_L(r) = \frac{1}{\gamma} \ln(L/r)}$$

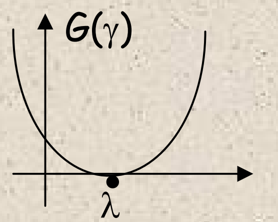
Link between exit-time $T_L(r)$ and finite time Lyapunov exponent γ

Statistics of vorticity

•Vorticity structure functions: $S_p^\omega(r) \equiv \left\langle \left| \delta\omega\ (r) \right|^p \right\rangle ; \ \left\langle \Omega^p \right\rangle_f \left\langle \left(\frac{r}{L} \right)^{\frac{p\alpha}{\gamma}} \right\rangle_\gamma$

•Distribution of finite time Lyapunov exponent γ (at long times):

$$P(\gamma,t) \approx t^{1/2} e^{-G(\gamma)t}$$



G(γ) (Cramér function) has a quadratic minimum at γ=λ (Lyapunov exponent)

•Prediction for structure functions

$$S_p^\omega(r) \approx \left\langle \Omega^p \right\rangle \int d\gamma \left(\frac{r}{L} \right)^{\frac{p\alpha + G(\gamma)}{\gamma}} \approx \left(\frac{r}{L} \right)^{\zeta(p)}$$

$$\zeta(p) = \min_\gamma \left\{ \left(p\alpha + G(\gamma) \right) / \gamma \right\}$$

* scaling exponents depend on friction coefficient α

* intermittency: nonlinear ζ(p) for generic G
for quadratic $G(\gamma)=(\gamma-\lambda)^2/2\mu$ one has the explicit expression

$$\zeta(p) = \frac{1}{\mu}\left[\sqrt{\lambda^2 + 2p\alpha\mu} - \lambda \right]$$

Correction to spectral slope

$$E(k) \approx k^{-3-\zeta(2)}$$

velocity is smooth

From passive to active problem

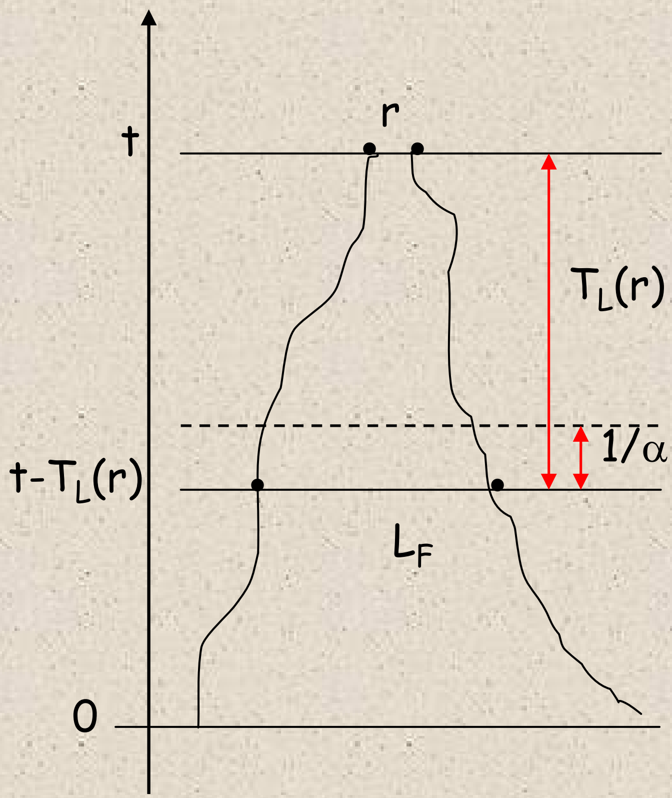
(if $f_\theta = f_\omega$, $\theta \to \omega$)

Crucial point: independence of forcing Ω and trajectories $T_L(r)$

$$\left\langle \Omega^p e^{-\alpha p T_L(r)} \right\rangle \overset{?}{=} \left\langle \Omega^p \right\rangle_f \left\langle e^{-\alpha p T_L(r)} \right\rangle_{T_L(r)}$$

Forcing of passive scalar do not affects velocity field
Forcing of vorticity changes both vorticity and trajectories

- Ω arises from forcing at time $0 < s < t - T_L(r)$
- $T_L(r)$ is determined by velocity at $t - T_L(t) < s < t$



Friction ➡ velocity has decorrelation time $1/\alpha$

- Ω and $T_L(r)$ are independent if $T_L(r) >> 1/\alpha$

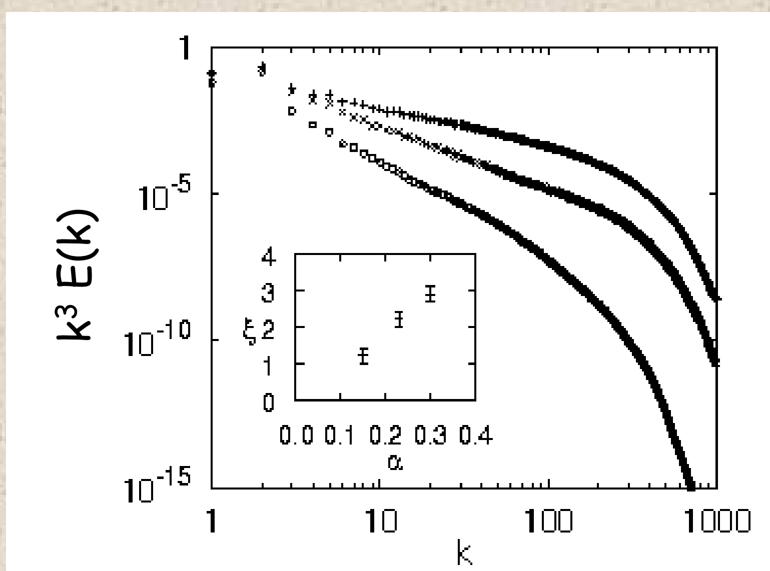
- In a smooth flow $T_L(r) = \frac{1}{\gamma} \ln(L_F / r)$ Passive condition: $\boxed{r << L_F \exp(-\gamma/\alpha)}$

Identical behaviour for vorticity and passive scalar at small scales

In non-smooth flow (e.g. Kolmogorov: $\delta u \approx r^{1/3}$) T_L is r-independent (dominated by large scales) thus passive condition is never satisfied.

Numerical results: effects of friction on spectrum

Energy spectrum always steeper than k^{-3}: smooth velocity field



$$E(k) \approx k^{-3-\zeta(2)}$$

Correction to spectral slope depends on friction intensity

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$\alpha=0.05$ Vorticity fields $\alpha=0.15$

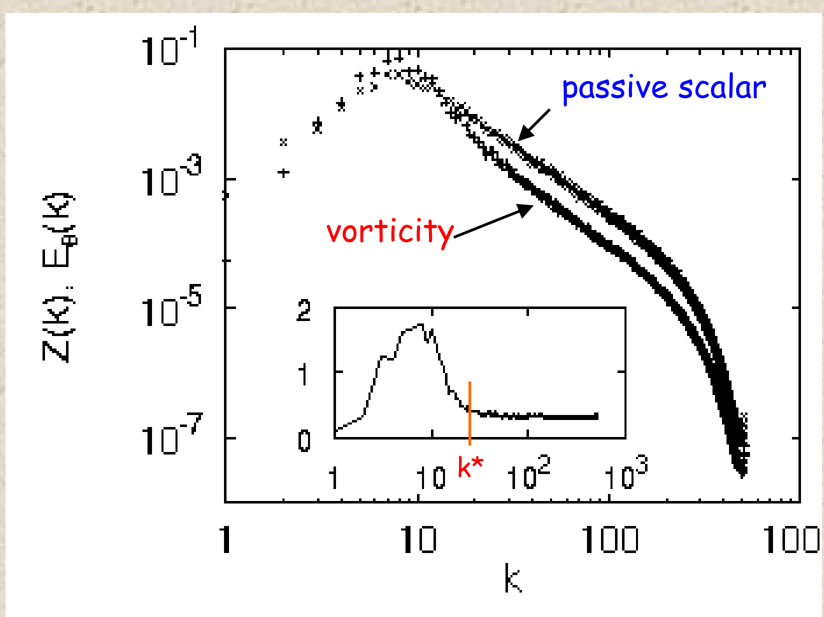
Comparing vorticity and passive scalar

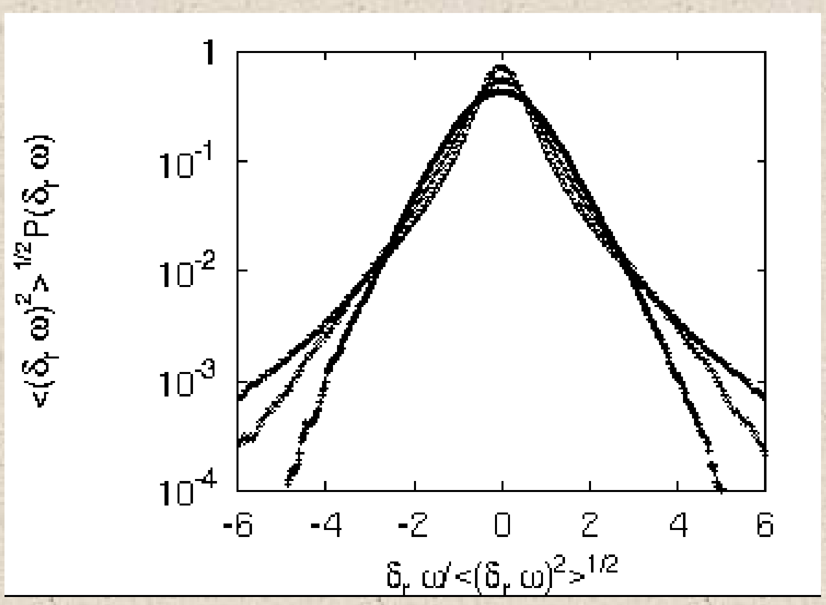
Power spectra of
vorticity (+) $Z(k)$
passive scalar (*) $E_\theta(k)$

Identical slope
$k^{-1-\zeta(2)}$
for $k \gg k^*$

passive condition

$$k ? \;\; k^* = k_F \exp(\lambda / \alpha)$$





Intermittency

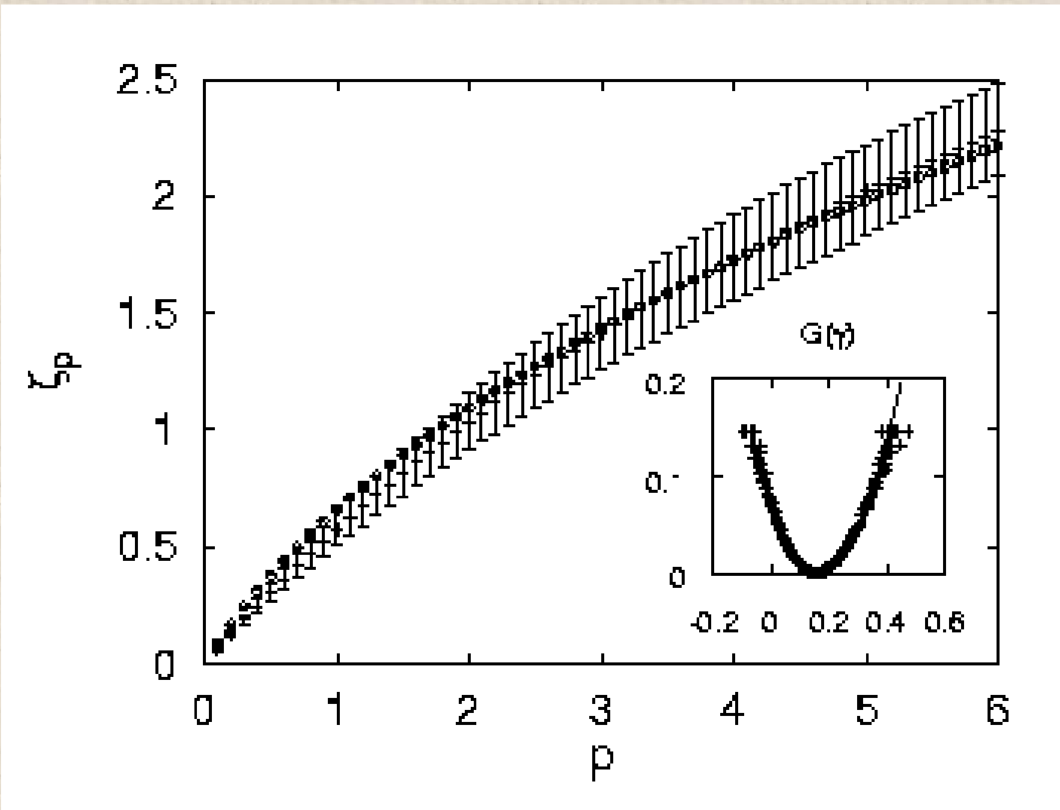
Probability density functions
$p(\delta\omega(r))$ not self-similar in r

Identical mechanism for
vorticity and passive scalar:
fluctuations of γ

Intermittency: anomalous scaling of vorticty structure functions

Prediction in terms of Cramer function

$$\zeta(p) = \min_{\gamma} \left\{ \left(p\alpha + G(\gamma) \right) / \gamma \right\}$$

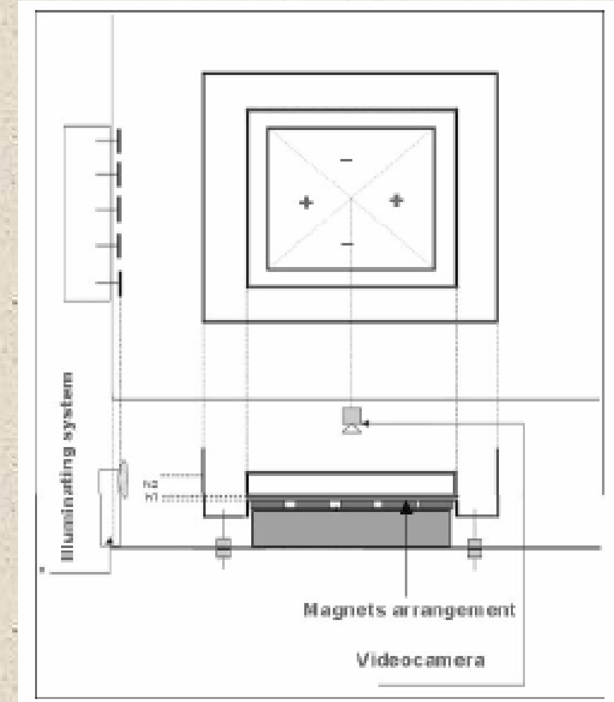


direct computation from SF

prediction from Cramer function

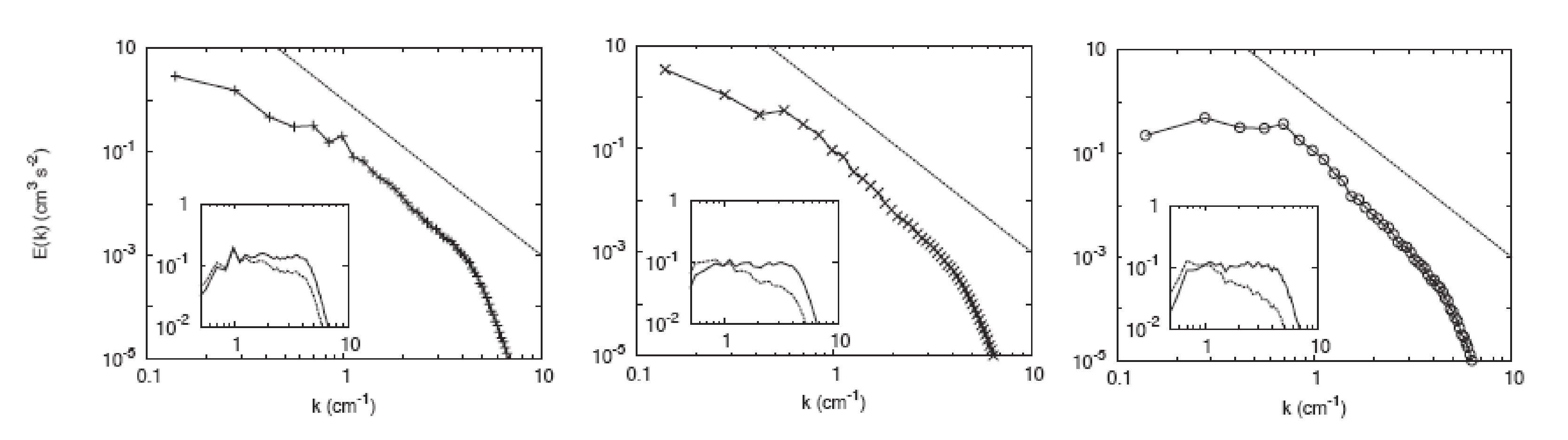
Experimental study of direct cascade with friction

G. Boffetta, A. Cenedese, S. Espa, S. Musacchio
Europhysics Letters **71**, 590 (2005).



Correction to the spectral slope observed
in experiments with electrolyte cell

$\alpha \approx \nu / h^2$



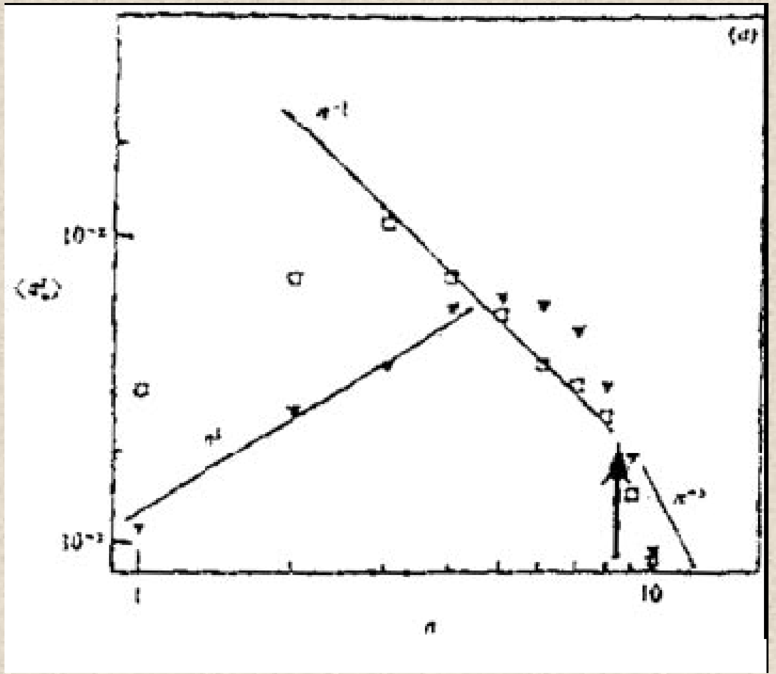
h=1cm h=0.9cm h=0.8cm

Inverse cascade

* Kraichnan-Kolmogorov spectrum $k^{-5/3}$ well observed
* absence of intermittency
* quasi-Gaussian statistics

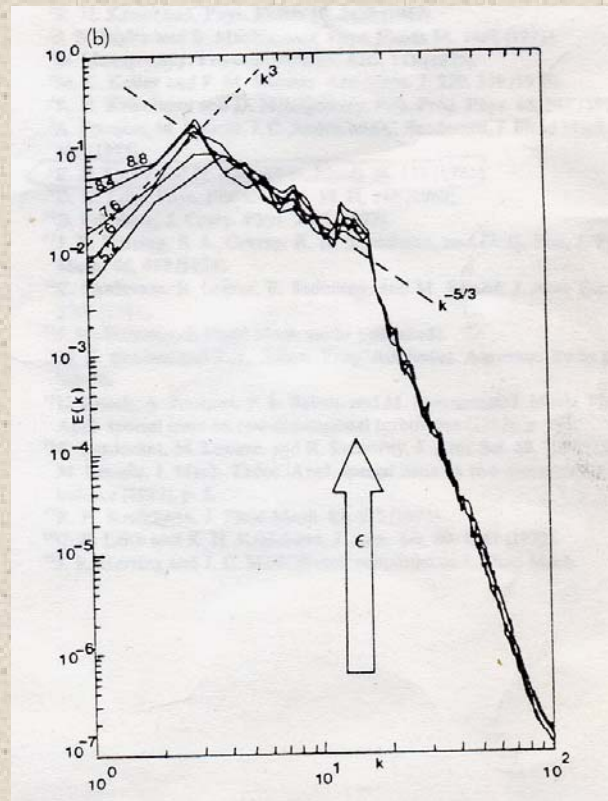
QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

Early studies on inverse cascade



Thin layer of mercury with electromagnetic forcing

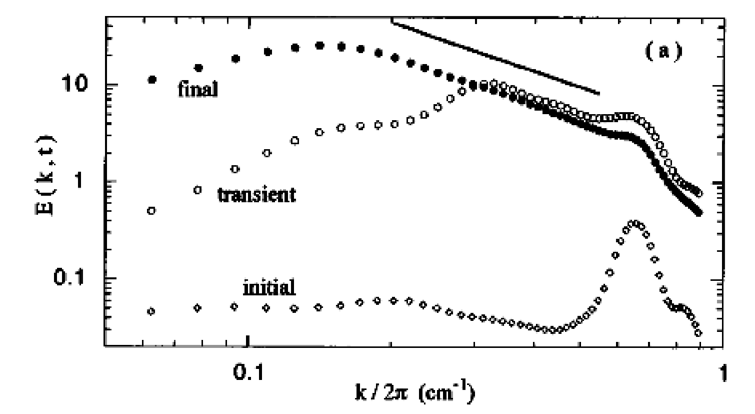
J.Sommeria, JFM **170**, 139 (1986)



Pseudo-spectral DNS at resolution 256^2

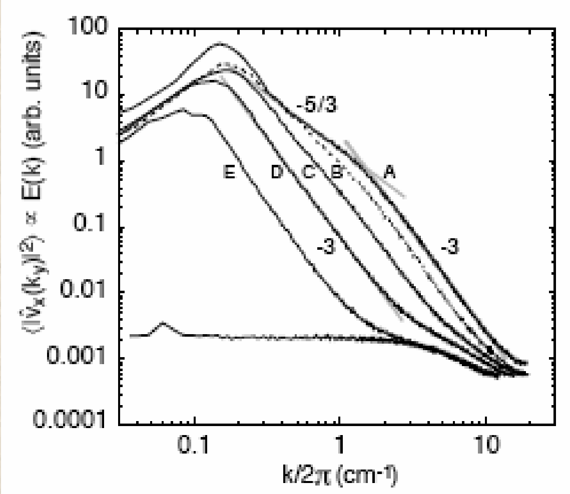
U.Frisch, P.L. Sulem, Phys. Fluids **27**, 1921 (1984)

More recent experiments ...



Electrolyte cell

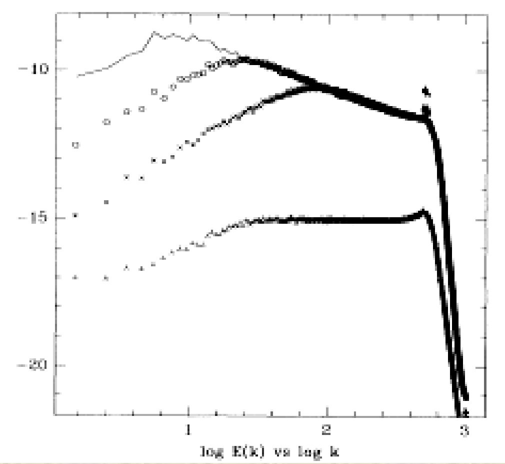
J. Paret, P.Tabeling, PRL **79** 4162 (1997)



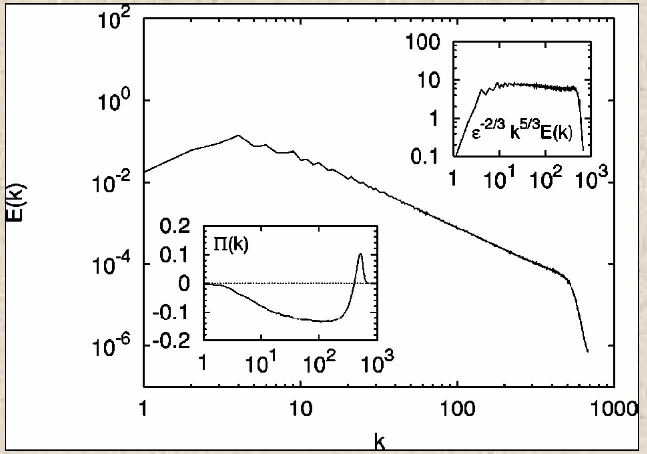
Soap film experiments

M.A. Rutgers, PRL **81**, 2244 (1998)

... and numerical simulations



L.M.Smith, V.Yakhot, PRL **71** 352 (1993)



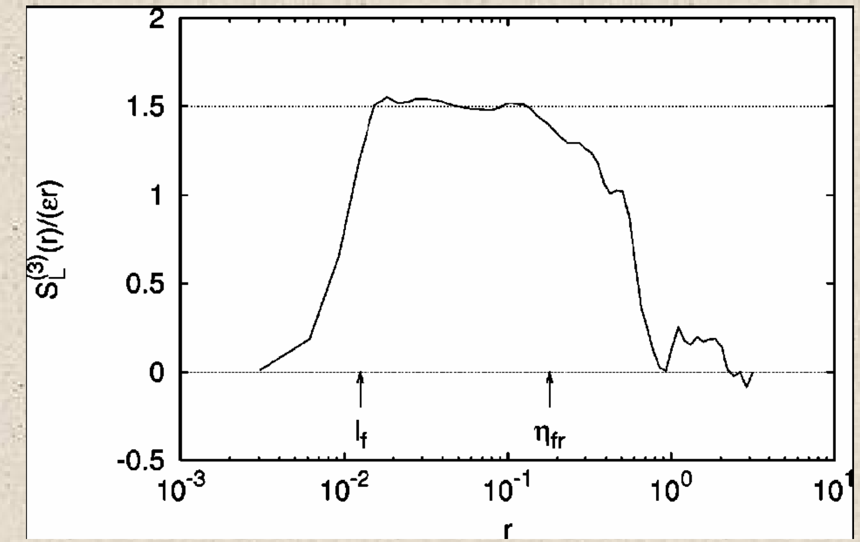
G.Boffetta, A.Celani, M.Vergassola, PRE **61** R29 (2000)

much wider inverse cascade range than experiments!

Kolmogorov law in inverse cascade (constant energy flux)

$$\left\langle \left(\delta u_{//}(r)\right)^3 \right\rangle = \frac{3}{2}\varepsilon r$$

between forcing scale r_F and friction scale $r_{fr}=\varepsilon^{1/2}\alpha^{-3/2}$

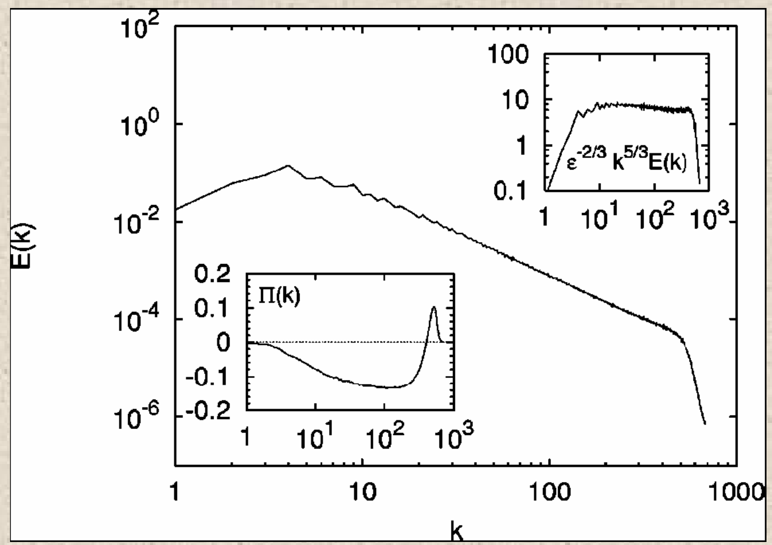


Energy spectrum and spectral flux

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

$$C = 6.0 \pm 0.4$$

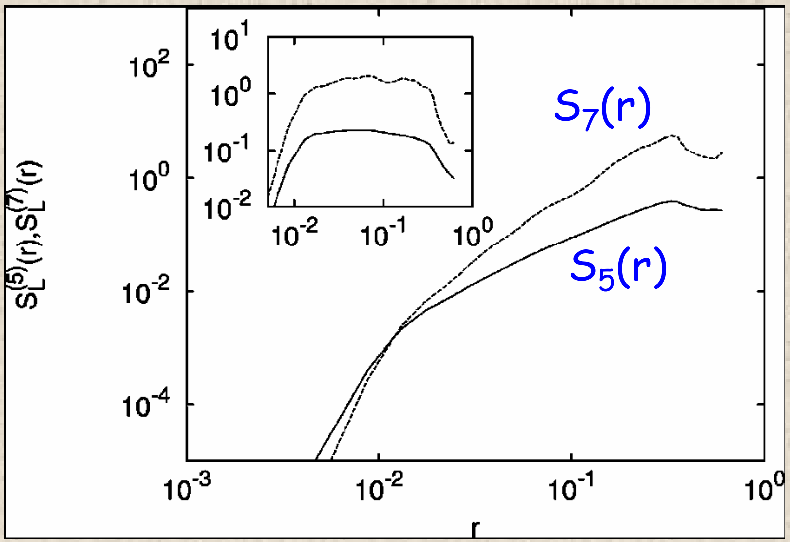
$$\left\langle \left(\delta u_{//}(r)\right)^2 \right\rangle ; \; 12.9(\varepsilon r)^{2/3}$$



Higher order structure functions

$$S_p(r) = \left\langle \left(\delta u_{//}(r)\right)^p \right\rangle = C_p \left(\varepsilon r\right)^{p/3}$$

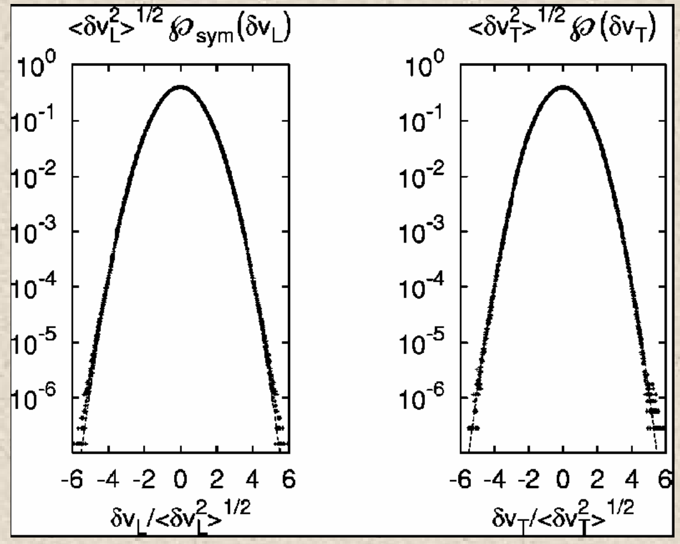
compatible with Kolmogorov scaling
no intermittency



Skewness is small (smaller than in 3d):

$$S = \frac{\left\langle \delta u(r)^3 \right\rangle}{\left\langle \delta u(r)^2 \right\rangle^{3/2}} \approx 0.03$$

pdf of velocity fluctuations
not far from Gaussian
(compared with 3d)



Deviations from Gaussian pdf

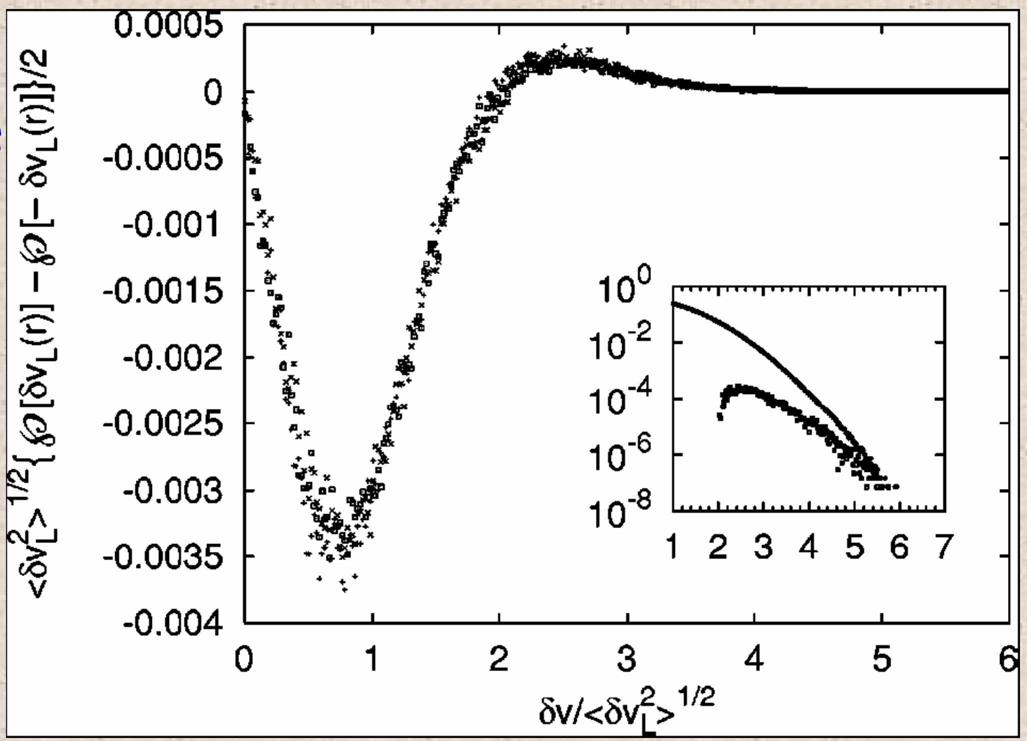
Antisymmetric part of the pdf:

$$p_A(u)=p(u)-p(-u)$$

is a measure of the deviation from Gaussian distribution

Antisymmetric part of longitudinal velocity increment pdf at separations r=0.05, 0.075, 0.1 (rescaling)

Inset: antisymmetric part (lower) compared with symmetric part (upper) at r=0.1



Core of the distribution not far from Gaussian, but very large deviations for large fluctuations δu (to be included in closures?)

Experimental study of two-dimensional enstrophy cascade
G. Boffetta, A. Cenedese, S. Espa and S. Musacchio
Europhysics Letters **71**, 590 (2005).

Intermittency in two-dimensional Ekman-Navier-Stokes turbulence
G. Boffetta, A. Celani, S. Musacchio and M. Vergassola,
Physical Review E **66**, 026304 (2002).

Closure of two dimensional turbulence: the role of pressure gradients
G. Boffetta, M. Cencini and J. Davoudi, *Physical Review E* **66**, 017301 (2002).

Inverse energy cascade in two-dimensional turbulence:
Deviations from Gaussian behavior
G. Boffetta, A. Celani and M. Vergassola, *Physical Review E* **61**, R29 (2000).

http://www.ph.unito.it/~boffetta

Statistical description

<...> ensemble averaging

Two point correlations $C_{\alpha\beta}(\mathbf{r}) = < u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}+\mathbf{r}) > = \sum_k e^{-i\mathbf{kr}} \hat{C}_{\alpha\beta}(\mathbf{k})$

Structure functions: $S_{\alpha\beta}(\mathbf{r}) = 2C_{\alpha\beta}(0) - 2C_{\alpha\beta}(\mathbf{r}) = \left\langle \delta_r u_\alpha \delta_r u_\beta \right\rangle$

Using incompressibility and isotropy: $\hat{C}_{\alpha\beta}(\mathbf{k}) = B(k)\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right]$

energy spectrum: $E(k) \equiv 2\pi k^2 B(k)$

Power-law energy spectrum

In scaling invariant inertial ranges one expects

$$E(k) \approx k^{-\gamma}$$

for $1 \leq \gamma \leq 3$ (locality) one has

$$S_{\alpha\alpha}(r) \sim r^{\gamma-1}$$

Higher order structure functions

$$S_p(r) \equiv \left\langle \left(\delta_r \mathbf{u} \cdot \hat{\mathbf{r}}\right)^p \right\rangle$$

$$\delta_r \mathbf{u} \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$$