Four-point correlation function in Minimal Liouville Gravity ①

(Al. Zamolodchikov, A.B.; hep-th/0510214)

Minimal Liouville Gravity – 2D quantum gravity induced by Minimal Conformal Matter.
It is a special case of Polyakov's String (A.P. 1981)

We get the explicit expression for the 4-point corr. function in a special case

Some relevant steps:

Conformal Bootstrap	– A.P. 1970
Noncrit. String, LFT	– A.P. 1981
C.T. Canon. Quant-n LFT, Spectr.	– C.T. 1982
Minimal Models of CFT	– B.P.Z. 1984
Light cone gauge Appr. and Spectrum of gravit. dim.s	– A.P. + KPZ 1988
	– DDK 1989
Gauge Invar. version of KPZ	– KPW 1991
Ground Ring structure	– DOZZ 1992–95
Exact solution of LFT	
Higher Eq-s of Motion in LFT	– Al. Z 2002

DDK formulation

Quantization problem: $\mathcal{D}^{Pol}[\varphi]$ - nonlinear measure

DDK conjecture (after KPZ):

Polyakov's approach is equivalent to the foll.

1. $\mathcal{D}^{Pol}[\varphi] \Rightarrow \mathcal{D}[\hat{g},\varphi]$ - ordinary linear measure
   $\hat{g}$ - background, gauge fixing metric

2. $A^{Pol}_{tot} \Rightarrow A_{tot} = A_M[\hat{g}, X_M] + A_L[\hat{g},\varphi] + A_{gh}[\hat{g},B,C]$

   A_{gh} - F-P action, A_M - conf. matter act.

$$A_L[\hat{g},\varphi] = \int \frac{1}{4\pi}\left[\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\varphi\hat{R} + \mu e^{2b\varphi}\right]\sqrt{\hat{g}}\,d^2x$$

– the modified Liouv. action; parameters Q, b - fixed by DDK consistency requir-s:

a. $A_L[\hat{g},\varphi]$ - conformal invariant

b. Total effective action - independent of gauge ($=$ of $\hat{g}_{ab}$)

Then

$$(a) \Rightarrow \quad Q = b + b^{-1}, \quad c_L = 1 + 6Q^2$$

$$(b) \Rightarrow \quad c_{tot} = c_M + c_L + c_{gh} = 0$$

Physical fields in $\mathcal{L}G$

Since the theory is $\hat{g}$-independent $\hat{g}_{ab} \Rightarrow \delta_{ab}$ (for sphere). Then

$$A_L[\varphi] = \int \left[\frac{1}{4\pi}(\partial_a \varphi)^2 + \mu e^{2b\varphi}\right] d^2x$$

$$A_{gh}[B,C] = \frac{1}{\pi}\int [C\bar{\partial}B + \bar{C}\partial\bar{B}]\, d^2x$$

(B,C) - ghosts of spin $(2,-1)$

$\langle \Psi_1 \ldots \Psi_N \rangle$ - corr. function of $_{\text{ph.f-s}}$

Ψ - physical field = gauge independent f. = BRST invariant field i.e.

$$\hat{Q}\Psi = 0$$

$$\hat{Q} = \oint [(T_M + T_L)C + :C\partial C\, B:]\, dz$$

The simplest case - Tachyon

$$W = \Phi_\Delta \cdot V_{\Delta'}\, \bar{C}C \quad , \quad \Delta + \Delta' = 1$$

Φ_Δ - prim. of Matter sector

$V_{\Delta'}$ - prim. of Liouv. sector

$$\hat{Q}W = 0$$

Generalized Minimal Models

depends on parametr b^2. (M_{b^2})

In standart MM $b^2 = p/p'$,

Finite set of degenerate primaries

$\{\Phi_{mn}, \Delta^M_{mn}\}$

GMM M_{b^2} – formal CFT

b^2 – continuos

$$C_M = 1 - 6(b^{-1} - b)^2$$

Spectrum of prim-s – continous

$\{\Phi_\alpha, \Delta^M_\alpha\}$, $\Delta^M_\alpha = \alpha(\alpha - q)$, $q = b^{-1} - b$

if $\alpha_{mn} = q/2 \pm \lambda_{mn}$, $\lambda_{mn} \overset{\text{def}}{=} \dfrac{nb + mb^{-1}}{2}$

then $\Phi_{\alpha_{mn}} = \Phi_{mn}$ – degenerate field

$$D^M_{mn} \Phi_{mn} = \bar{D}^M_{mn} \Phi_{mn} = 0$$

D^M_{mn} – singular vector creating operator made of Vir generators

$$D^M_{12} = M^2_{-1} - b^2 M_{-2}$$

$$\{\Phi_{mn}\} \subset \{\Phi_\alpha\}$$

OPE of the degenerate fields – known

Liouville Field Theory

LFT is solved in principal (DOZZ)

1. LFT - nonrational CFT

$$c_L = 1 + 6Q^2 \,, \quad Q = b + b^{-1}$$

(if b - the same as for $M_{b^2} \Rightarrow c_M + c_L = 26$)

primaries: $V_a(z) = \exp 2a\varphi(z) \,, \; \Delta_a^L = a(Q-a)$

2. OPE is explicitly known (ZZ, 1995)

$$V_{a_1}(x) V_{a_2}(0) = \int_C \frac{dP}{4\pi} C_{a_1,a_2}^{P} [V_P]$$

$$C_{a_1 a_2}^{a_3} = C_{a_1,a_2,Q-a_3} \quad \text{(DOZZ)}$$

$$C_{a_1 a_2 a_3} = \left[\pi \mu \gamma(b^2) b^{2-2b^2}\right]^{\frac{Q-a}{b}} \frac{\Upsilon(b)}{\Upsilon(a-b)} \prod_{i=1}^{3} \frac{\Upsilon(2a_i)}{\Upsilon(a-2a_i)}$$

$$a = \sum_{i=1}^{3} a_i \,, \quad \gamma(x) = \Gamma(x)/\Gamma(1-x)$$

$\Upsilon(x)$ - Barnes function

3. If $a = a_{mn} = Q - \lambda_{mn}$, then

$V_{mn} \stackrel{def}{=} V_{a_{mn}}$ - degenerate field.

$$D_{mn}^L V_{mn} = \bar{D}_{mn}^L V_{mn} = 0$$

$\{V_{mn}\} \subset \{V_a\}$ - this fact + assoc-ty OPE imply the form $C_{a_1 a_2 a_3}$

Physical Fields in MLG

Let $U_a \overset{\text{def}}{=} \Phi_\alpha V_a$, $\Delta^M_\alpha + \Delta^L_a = 1 \Rightarrow \alpha = a - b$

i.e. $U_a = \Phi_{a-b} V_a$ – (1,1) form, $N_{gh} = 0$

$\hat{Q} U_a = \partial(B_{-1} U_a) \Rightarrow \int U_a d^2x$ – gauge inv-t

$W_a \overset{\text{def}}{=} \bar{C} C U_a$ – (0,0) form of $N_{gh} = 1$

$Q W_a = 0$, W_a – gauge inv-t

$\partial W_a = 0 \mod Q$

"Discrete states" (KPW)

$O_{mn} = \bar{H}_{mn} H_{mn} \Phi_{mn} V_{mn}$, $N_{gh}(O_{mn}) = 0$

H_{mn} – graded polyn-l of M_n, L_n, B, C defined by

$\hat{Q} O_{mn} = 0$

Example. $H_{12} = M_{-1} - L_{-1} + b^2 : CB :$

$\partial O_{mn} = 0 \mod Q$

The most general correlator (Veneziano ampl.):

$$\langle\langle U_{a_1} \ldots U_{a_N} O_{m_1 n_1} \ldots O_{m_K n_K} \rangle\rangle \overset{\text{def}}{=}$$

$$= \int \langle W_{a_1}(x_1) W_{a_2}(x_2) W_{a_3}(x_3) U_{a_4}(x_4) \ldots U_{a_N}(x_N) O_{m_1 n_1}(z_1) \ldots O_{m_K n_K}(z_K) \rangle \prod_{i=4}^{N} d^2 x_i$$

The simplest case: $\langle\langle U_{a_1} U_{a_2} U_{a_3} \rangle\rangle = \langle W_{a_1} W_{a_2} W_{a_3} \rangle =$

$$= \Omega \prod_{i=1}^{3} N(a_i); \qquad \Omega = [\pi \mu \gamma(b^2)]^{Q/b} \left[-\frac{\gamma(b^2)\gamma(b^{-2})}{(b^{-1}-b)^2} \right]^{1/2},$$

$$N(a) = [\pi \mu \gamma(b^2)]^{-\frac{a}{b}} \left[\gamma(2ab - b^2) \gamma(2a/b - b^{-2}) \right]^{1/2}$$

Ground Ring in MLG (SS)

$\{W_a, O_{mn}\}$ exhast all BRST closed f. for generic b^2 and, as cohomologies, form Ring (due to OPE)

1. $$\boxed{O_{mn}\hat{W}(a) = \Delta_{mn}\sum_{r,s\in(m,n)_2}\hat{W}(a+\lambda_{rs})}$$

$$\hat{W}(a) \stackrel{\text{def}}{=} W_a / N(a)$$

$$(m,n)_2 \stackrel{\text{def}}{=} (\{1-m, 3-m, \ldots, m-1\}, \{1-n, 3-n, \ldots, n-1\})$$

$$\boxed{\Delta_{mn} = \left[\frac{\gamma(b^2)\gamma(b^{-2})}{(b-b^{-1})^2}\right]^{1/2} B_{mn} N(a_{m,-n})}$$

B_{mn} - defined on the next page

2. $\{O_{mn}\}$ form subRing (= Ground Ring of KPW)

$$\boxed{\hat{O}_{m_1n_1}\hat{O}_{m_2n_2} = \sum_{n\in[n_1,n_2]}\sum_{m\in[m_1,m_2]}\hat{O}_{mn}}$$

$$\hat{O}_{mn} \stackrel{\text{def}}{=} O_{mn}/\Delta_{mn}$$

$$[n_1, n_2] \stackrel{\text{def}}{=} |n_1-n_2|+1, |n_1-n_2|+3, \ldots, n_1+n_2-1$$

Higher Equations of Motion in LFT (Al.Z) ⑧

Let $V'_{mn} \overset{\text{def}}{=} \frac{1}{2}\frac{d}{da}V_a\Big|_{a=a_{mn}}$ – logar. deg. field

$$\left|\frac{dy}{dx}\right|^{2\Delta^L_{mn}} V'_{mn}(y(x)) = V'_{mn}(x) - \Delta^{L\prime}_{mn}\log\left|\frac{dy}{dx}\right| V_{mn}(x)$$

$$\boxed{\bar{\mathcal{D}}^L_{mn}\mathcal{D}^L_{mn}V'_{mn} = B_{mn}\tilde{V}_{m,n}}$$

$\tilde{V}_{m,n} = V_{m,-n}$ – primary L. field $\tilde{\Delta}^L_{m,n} = 1 - \Delta^M_{mn}$

Very important for MLG: $\tilde{V}_{mn}$ - dressing Φ_{mn}

i.e. $U_{mn} = \Phi_{mn}\tilde{V}_{mn}$ - (1,1) form

$$B_{mn} = [\pi\mu\gamma(b^2)]^n b^{1+2n-2m}\gamma(m-nb^2)\prod_{r,s\in(m,n)}(2\lambda_{rs})$$

$(m,n) \overset{\text{def}}{=} (\{1-m, 2-m, \dots, m-1\}, \{1-n, \dots, n-1\}) / (0,0)$

It follows from HEM (Al.Z):

$$\boxed{\bar{\partial}\partial O'_{mn} = B_{mn}U_{mn} \mod Q}$$

$$O'_{mn} \overset{\text{def}}{=} \bar{H}_{mn}H_{mn}\Phi_{mn}V'_{mn}$$

It follows from OPE:

$$\boxed{O'_{mn}(x)\hat{W}_a(0) \underset{x\to 0}{\Rightarrow} \ln(\bar{x}x)\sum_{r,s\in(m,n)_2} q_{mn}(a-\lambda_{rs})\hat{W}(a-\lambda_{rs}) + \text{less singular terms}, (\text{mod } \hat{Q})}$$

where $q_{mn}(a) = |a - Q/2| - \lambda_{mn}$

4-point correlator

$$\langle\langle U_{mn} U_{a_1} U_{a_2} U_{a_3} \rangle\rangle = \int d^2x \, \langle U_{mn}(x) W_{a_1}^{(x_1)} W_{a_2}^{(x_2)} W_{a_3}^{(x_3)} \rangle$$

$$\text{HEM} \Longrightarrow U_{mn} = B_{mn}^{-1} \bar{\partial}\partial O'_{mn} \quad \text{mod } \hat{Q}$$

Integrating by parts:

$$\langle\langle U_{mn} U_{a_1} U_{a_2} U_{a_3} \rangle\rangle =$$

$$= B_{mn}^{-1} \Big\{ \lim_{R\to\infty} \oint_{C_R} \frac{dz}{2i} \frac{\partial}{\partial z} \langle O'_{mn}(z) \prod_{k=1}^{3} W_{a_k}(x_k) \rangle -$$

$$- \sum_{i=1}^{3} \lim_{r\to 0} \oint_{C_r^i} \frac{dz}{2i} \frac{\partial}{\partial z} \langle O'_{mn}(z) \prod_{k=1}^{3} W_{a_k}(x_k) \rangle \Big\}$$

From the transform. law for O'_{mn}

$$\left(O'_{mn}(y) = O_{mn}(x) - \Delta'_{mn} O_{mn}(x) \log |y_x| \right)$$

follows that

$$\langle O'_{mn}(x) \prod_{k=1}^{3} W_{a_k}(x_k) \rangle \underset{x\to\infty}{\Longrightarrow} -2\Delta'_{mn} \log(\bar{x}x) \langle O_{mn} \prod_{k=1}^{3} W_{a_k}(x_k) \rangle$$

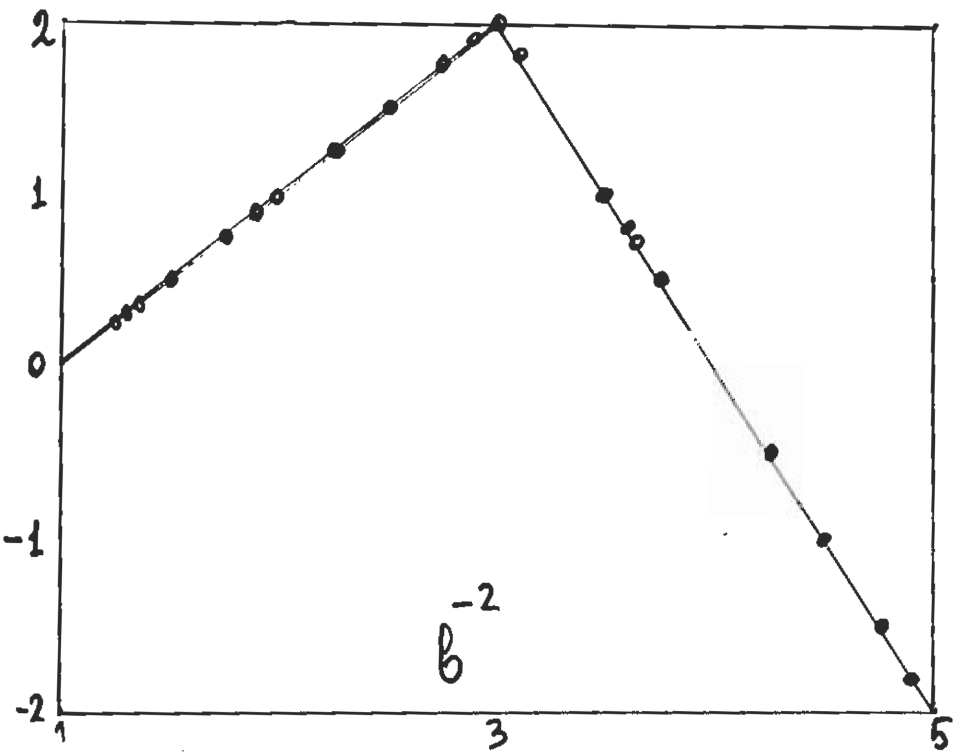
Together with $O'_{mn}(x) W_a(x_i)$ exp-n result to

$$\boxed{\begin{aligned} &\langle\langle U_{mn} U_{a_1} U_{a_2} U_{a_3} \rangle\rangle = \pi \left[\pi \mu \gamma(b^2)\right]^{Q/b} \frac{\gamma(b^2)\gamma(b^{-2})}{(b-b^{-1})^2} \Sigma_{mn}(a_1 a_2 a_3) \\ &\Sigma_{mn}(a_1 a_2 a_3) \overset{\text{def}}{=} mn\lambda_{mn} - \sum_{i=1}^{3} \sum_{r,s\in(m,n)_2} |\lambda_i - \lambda_{rs}|_{Re} \\ &\quad \text{where } \lambda_i \overset{\text{def}}{=} Q/2 - a_i \end{aligned}}$$

Numerical check: direct integration

Properly normalized

$$\langle\langle V_{12}V_{12}V_{12}V_{12}\rangle\rangle \Rightarrow 7/2 - \tfrac{1}{2}b^{-2} - \tfrac{3}{2}|b^{-2}-3|$$



Comparing with matrix models

in cases when it is possible
confirms the 4-p expression :
$M(3/4)LG$, $M(2/5)LG$ and
the gravitational $O(n)$ model (Kostov)
where our expression $\Rightarrow$

$$\langle\langle V_{13}V_{13}V_{13}V_{13}\rangle\rangle \approx \tfrac{3}{2}\left[3+b^{-2}-|b^{-2}-1|-|b^{-2}-3|-|b^{-2}-5|\right] \Rightarrow 3(b^{-2}-3)$$

for $1<b^{-2}<3$

coincides with Kostov's result.